October 7

Wednesday, October 7, 2020 4:04 PM

Comp Sci 212

1. Binomial theorem
2. Principle of Inclusion-Exclusion

Announcements

- Grading questions
- Getting to know you meetings

Theorem. # of subsets of size ℓ of a set S, s.t. $|S| = k$ is $\frac{k!}{\ell!(k-\ell)!} = \binom{k}{\ell}$

"k choose ℓ"

Theorem. $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$. $(*)$

Proof. Use bijection rule. Let

A = set of subsets of $[n+1]$ w/ k elements.

$[n+1] = \{1, 2, \ldots, n+1\}$

A_1 = set of subsets of $[n]$ w/ k elements.

A_2 = set of subsets of $[n]$ w/ $k-1$ elements.

$|A| = \binom{n+1}{k}$ $\quad |A_1| = \binom{n}{k}$ $\quad |A_2| = \binom{n}{k-1}$

Need to prove, $|A| = |A_1| + |A_2|$.

$f: A \to A_1 \cup A_2$.

$$f(x) = \begin{cases} x & \text{if } x \text{ does not contain } n+1 \\ x \setminus \{n+1\} & \text{o.w.} \end{cases}$$

$n=6, k=3$ $\quad f(\{1,4,7\}) = \{1,4\}$

$f(\{1,4,6\}) = \{1,4,6\}$. $\quad \{1,4,8\} \notin A$

$$f^{-1}(x) = \begin{cases} x \cup \{n+1\} & \text{if } |x| = k-1 \\ x & \text{if } |x| = k \end{cases}$$

$|f^{-1}(x)| = 1$ for all $x \in A_1$, $x \in A_2$, so f is a bijection, and $|A| = |A_1 \cup A_2|$.

Finally, by the sum rule, $|A_1 \cup A_2| = |A_1| + |A_2|$.

$(A_1 \cap A_2 = \emptyset)$.

$\square$

"To choose k elements from $[n+1]$, either choose $k-1$ elements from $[n]$ and $n+1$, or choose k elements from $[n]$."

Def. $\binom{n}{k} = 0$ if $k \leq -1$, or $k \geq n+1$.

$\binom{n}{0}x^0 + \binom{n}{1}x^1 + \binom{n}{2}x^2 \ldots + \binom{n}{n}x^n$

Binomial Theorem.

Thm. If $n \in \mathbb{N}$, $(1+x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i$

Proof. Use induction.

Base case $n=0$, $(1+x)^n = 1 = \binom{0}{0}x^0$

Inductive step. Assume $(1+x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i$

Then $(1+x)^{n+1} = (1+x)(1+x)^n$

$= (1+x)\left(\sum_{i=0}^{n} \binom{n}{i} x^i\right)$

$= \sum_{i=0}^{n} \binom{n}{i} x^i + \sum_{i=0}^{n} \binom{n}{i} x^{i+1} \longrightarrow \binom{n}{0}x + \binom{n}{1}x^2 + \ldots$

$= \sum_{i=0}^{n} \binom{n}{i} x^i + \sum_{i=1}^{n+1} \binom{n}{i-1} x^i \longrightarrow \binom{n}{0}x^1 + \binom{n}{1}x^2$

$= \sum_{i=0}^{n} \binom{n}{i} x^i + \sum_{i=1}^{n+1} \binom{n}{i-1} x^i + \binom{n}{n+1}x^{n+1} + \binom{n}{-1}x^0$

$= \sum_{i=0}^{n+1} \binom{n}{i} x^i + \sum_{i=0}^{n+1} \binom{n}{i-1} x^i = \sum_{i=0}^{n+1} \left(\binom{n}{i} + \binom{n}{i-1}\right) x^i \overset{(*)}{=} \sum_{i=0}^{n+1} \binom{n+1}{i} x^i$ $\square$

Corollary. If $n \in \mathbb{N}$, $(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$

Prof. By binomial, $\left(1 + \frac{a}{b}\right)^n = \sum_{i=0}^{n} \binom{n}{i} \frac{a^i}{b^i}$

Thus, $(a+b)^n = b^n \left(1 + \frac{a}{b}\right)^n$

$(a+b) = b\left(1 + \frac{a}{b}\right)$ $\quad = b^n \cdot \sum_{i=0}^{n} \binom{n}{i} \frac{a^i}{b^i}$

$= \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$

$\square$

Theorem. $\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$.

Prof. $\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = \sum_{i=0}^{n} \binom{n}{i} 1^i$

by binomial theorem. $= (1+1)^n$

$= 2^n$

Theorem. $\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \ldots =$

$\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \ldots$

Prof. $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \ldots$

$= \sum_{i=0}^{n} \binom{n}{i} (-1)^i$

$= (1 + (-1))^n$ (by binomial theorem)

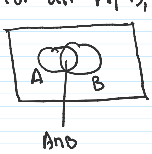
$= 0$.

$\square$

Principle of Inclusion-Exclusion

(sum rule update)

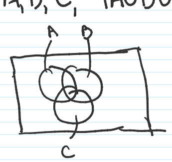
For all A, B, $|A \cup B| = |A| + |B| - |A \cap B|$



$|A| + |B|$ - counting elements of $A \cap B$ twice

A, B, C, $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B|$

$- |A \cap C| - |B \cap C| + |A \cap B \cap C|$



$|A \cap B \cap C|$

Ex. How many numbers from 1 to 100 are divisible by 2 or 5?

$= |A \cup B| = |A| + |B| - |A \cap B|$

$= |A| + |B| - |C|$.

$A = \{x \mid 1 \leq x \leq 100, x \in \mathbb{Z}, x/2 \in \mathbb{Z}\}$

Ex. How many numbers from 1 to 100 are divisible by 2 or 5?

$A = \{x \mid 1 \le x \le 100,\ x \in \mathbb{Z},\ x/2 \in \mathbb{Z}\}$

$B = \{x \mid 1 \le x \le 100,\ x \in \mathbb{Z},\ x/5 \in \mathbb{Z}\}$

$C = \{x \mid 1 \le x \le 100,\ x \in \mathbb{Z},\ x/10 \in \mathbb{Z}\}.$

$= |A \cup B| = |A| + |B| - |A \cap B|$

$= |A| + |B| - |C|.$

$|A| = 50,\ |B| = 20,\ |C| = 10$

$|A \cup B| = 50 + 20 - 10 = 60.$