October 5

Monday, October 5, 2020 3:58 PM

Comp Sci 212

1 Counting Rules ctd.

Announcements
- Plan for exams

Generalized Product Rule - If S is a set of length-k sequences $(s_1, s_2, s_3, \dots, s_k) \in S$
- n_1 = # of possibilities for s_1
- n_2 = # of possibilities for s_2

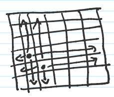
$\vdots$
- n_k = # of possibilities for s_k

$|S| = n_1 \cdot n_2 \cdots n_k$.

Ex. How many ways are there to place k distinguishable rooks on a $k \times k$ chessboard,

every row can have at most 1 rook

column can have at most 1 rook



Use generalized product rule

$n_1 = k^2$

$n_2 = (k-1)^2$

$n_3 = (k-2)^2$

$\vdots$

$n_i = (k-i+1)^2$

$\vdots$

$n_k = (k-k+1)^2 = 1$

# of ways $= n_1 \cdot n_2 \cdot n_3 \cdots n_k$

$= k^2 (k-1)^2 \cdot (k-2)^2 \cdots 1$

$= (k!)^2$

Def. Permutation - a length k sequence of elements from a set of size k, no two are the same.

$S = \{a, b, c, d\}$ ex. permutations - (c, b, a, d)

(a, b, c, d)

Theorem. # of permutations of a set S of size k is $k! = k \cdot (k-1) \cdot (k-2) \cdots 1$.

Proof. Use generalized product rule. $n_1 = k$, $n_2 = k-1$, $n_i = k-i+1$, so # of permutations $= n_1 n_2 \cdots n_k = k!$ □

Sorting input - permutation of set

output - sorted list

Merge-sort - list of size k, running time $O(k \log k)$.

Sorting algorithm - assigns a sequence of swaps to each permutation

length of sequence of swaps $\geq$ running time

must do something diff. for each permutation.

A = set of permutations - $|A| = k!$

B = set of sequences of swaps $\subseteq \{0,1\}^{\ell}$ running time

$|\{0,1\}^{\ell}| = 2^{\ell}$

$f: A \to B$, sorting algorithm

f needs to be injective $\to |A| \leq |B|$

$\to k! \leq 2^{\ell}$

$\ell = \Omega(n \log n)$

Division rule

Def. $f: A \to B$ is k-to-1 if $|f^{-1}(b)| = k$ for every $b \in B$.

Division rule - If $f: A \to B$ is k-to-1, then $|A| = k \cdot |B|$.

Theorem. # of length ℓ sequences of elements from S, s.t. (such that) $|S| = k$, no two the same, $\frac{k!}{(k-\ell)!}$

Proof. A = set of permutations of S.

B = set of length-ℓ sequences from S.

$f: A \to B$, $f(a)$ = first ℓ elements of a

$k = 6, \ell = 2$

$a = (6, 1, 2, 3, 4, 5)$, $f(a) = (6, 1)$

Elements of $f^{-1}(b) \to b$ concatenated w/ permutation of $S \setminus b$ view as a set

$|f^{-1}(b)|$ = # of permutations of $S \setminus b$, $|S \setminus b| = (k-\ell)!$, f is $(k-\ell)!$-to-1.

By division rule $|B| = \frac{|A|}{(k-\ell)!} = \frac{k!}{(k-\ell)!}$

Thm. # of subsets of size ℓ of S, $|S| = k$ is $\frac{k!}{(k-\ell)!\,\ell!} = \binom{k}{\ell}$

Proof. A = set of length ℓ sequences from S

B = set of subsets of S of size ℓ.

$f: A \to B$, $f(a)$ = a viewed as a set

$f^{-1}(b)$ = set of permutations of b

$k = 6, \ell = 2$

$f((6,1)) = \{1, 6\}$

$f((1,6)) = \{1, 6\}$

f is $\ell!$-to-1 ($|f^{-1}(b)| = \ell!$), by division rule, $|B| = \frac{|A|}{\ell!} = \frac{k!}{(k-\ell)!\,\ell!}$

Thm. # of sequences of length-10 w/ 2 1's $= \{0,1\}^{10}$

# of subsets of $[10] = \{1, 2, \dots, 10\}$ of size 2.

Proof. B = set of sequences of length 10 w/ 2 1's ($\subseteq \{0,1\}^{10}$)

C = set of subsets of $[10]$ of size 2.

$f: B \to C$ $\quad f(b) = \{i \mid b_i = 1, i \in [10]\}$

$(0,1,0,0,0,0,1,0,0,0) = b \quad f(b) = \{2, 7\}$.

$c \in C$, $c = \{c_1, c_2\} \subseteq [10]$,

$f^{-1}(c) = (b_1\, b_2\, b_3, \dots, b_{10})$ s.t. $b_{c_1}, b_{c_2} = 1$, rest 0.

$|f^{-1}(c)| = 1$, injective, surjective function, total $\to$ bijection, $|B| = |C|$ by bijection rule.

Corollary. # of ways to buy 8 donuts 3 types is

# of ways to buy n donuts of k types =

# of 0/1 strings of length $n+k-1$ w/ $k-1$ 1's =

# of subsets of size $k-1$ of a set of size $n+k-1$ =

$\binom{n+k-1}{k-1}$

function, total

by bijection rule.

Corollary. # of ways to buy 8 donuts 3 types is
$\binom{10}{2}$.

$C = \{1, 2\} \rightarrow$
$(1, 1, 0, 0, 0, 0, \ldots, 0)$