September 30

Wednesday, September 30, 2020 3:54 PM

Comp Sci 212

1. Functions
2. Asymptotics

Announcements

- Homework 2 out (list of things updated)
- Getting to know you meetings

Relation subset of $A \times B$ (domain, range)

If function $f: A \to B$, $(a, f(a)) \in$ subset

Inverse image $f^{-1}(b) = \{a \mid f(a) = b\}$

- injective if $f(a) = f(b)$ implies $a = b$
  $|f^{-1}(b)| \le 1$ for all $b \in B$
- surjective if $|f^{-1}(b)| \ge 1$ for all $b \in B$

Ex. $f: [n] \to [2n]$, $f(a) = 2a$ ($[n] = \{1, 2, 3, \dots, n\}$, $[2n] = \{1, 2, 3, \dots, 2n\}$)

$f$ is injective, $f(a) = f(b) \to 2a = 2b \to a = b$

$f$ is not surjective, $f^{-1}(b) = \phi$ if $b$ is odd $\in [2n]$

$(f: [n] \to \{2, 4, 6, \dots, 2n\}, \ f(a) = 2a, )$

then $f$ would be a surjection

Ex. $\{(a, b) \mid a \in [n],\ b \in \{\text{even}, \text{odd}\},\ a \text{ is "}b\text{"}\} \subseteq [n] \times \{\text{even}, \text{odd}\}$

$g: [n] \to \{\text{even}, \text{odd}\}$ (1, odd) (2, even)

$$g(a) = \begin{cases} \text{even if } a \text{ is even} \\ \text{odd if } a \text{ is odd} \end{cases}$$

If $n \ge 2$, $g$ is surjective, $1 \in g^{-1}(\text{odd})$, $2 \in g^{-1}(\text{even})$.

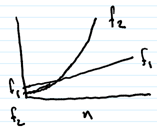
if $n \ge 3$. not injective, $1, 3 \in g^{-1}(\text{odd})$ $|g^{-1}(\text{odd})| \ge 2$

Asymptotics

Question - what is the behavior of $f: \mathbb{R} \to \mathbb{R}$ as $n$ becomes large.

- running time of algorithms
- used in other areas of math

$f_1(n) = 100 \cdot n \log n \quad f_2(n) = 2n^2 - n$




Def. $f(n) = O(g(n))$ if $\limsup_{n \to \infty} \left|\frac{f(x)}{g(x)}\right| < \infty$ (lim)

- g grows faster, or as fast as f

$f(n) = o(g(n))$ if $\limsup_{n \to \infty} \left|\frac{f(x)}{g(x)}\right| = 0$

~'s interpretations - g grows faster than f

$f(n) = \Omega(g(n))$ if $g(n) = O(f(n))$

- f grows faster or as fast as g

$f = \omega(g(n))$ if $g(n) = o(f(n))$

- f grows faster than g

$f(n) = \Theta(g(n))$ if $f(n) = O(g(n))$ & $g(n) = O(f(n))$

- g grows as fast as f.

- f grows faster $f = \Omega(g)$ $\omega(g)$
- f grows as fast as $f = \Omega(g)$ $O(g)$ $\Theta(g)$
- f grows slower $f = O(g)$ $o(g)$

Ex. $2n^2 + n = \Theta(n^2)$ $\lim_{n \to \infty} \frac{2n^2+n}{n^2} = 2$

$\lim_{n \to \infty} \frac{n^2}{2n^2+n} = \frac{1}{2}$

$n = o(n^2)$ $\lim_{n \to \infty} \frac{n}{n^2} = 0$

$1 = \omega\left(\frac{1}{\log n}\right)$ $\lim_{n \to \infty} \frac{\frac{1}{\log n}}{1} = 0$ $\left(\frac{1}{\log n} = o(1)\right)$

- Ignore constant factors
- Ignore lower order terms

Ex. $a_n x^n + a_{n-1} x^{n-1} + \dots + a_0 = \Theta(x^n)$

Fact. If $a < b$, then $\lim_{n \to \infty} \frac{n^a}{n^b} = 0$.

$(n^a = o(n^b))$

$\log_2(n) = \frac{\ln(n)}{\ln(2)}$

Thm. For any $a, b > 0$, $\lim_{n \to \infty} \frac{\log^a(n)}{n^b} = 0$.

$\log^{10000000}(n) = o(n^{0.0000001})$

Proof. Assume $a = 1$.

By l'Hôpital's rule, $\lim_{n \to \infty} \frac{\log(n)}{n^b} = \lim_{n \to \infty} \frac{\frac{1}{n}}{bn^{b-1}}$

$= \lim_{n \to \infty} \frac{1}{b} \cdot n^{-b} = 0$ if $b > 0$.

Thm. If $\log(f(n)) = o(\log(g(n)))$, $f(n), g(n) > 0$ for all $n$,

then $f(n) = o(g(n))$.

Ex. $f(n) = e^{\sqrt{n}}$, $g(n) = n^{100}$

$\log(f(n)) = \sqrt{n}$ $\quad \log(g(n)) = 100 \log(n)$

$\log(g(n)) = o(\log(f(n))) \to g(n) = o(f(n))$.

Ex. $f(n) = \log(n)^{\log(n)}$ $\quad g(n) = n^{100}$

Ex. $f(n) = (\log(\log(n)))^{\log(n)}$ $\quad g(n) = n^2$ ← looks like this grows slower on a graphing calculator

$\log(f(n)) = \log(n) \cdot \log(\log(\log(n)))$

$\log(g(n)) = 2\log(n)$

$\lim_{n \to \infty} \frac{\log(g(n))}{\log(f(n))} = \frac{2}{\log(\log(\log(n)))} = 0$

$\log(g(n)) = o(\log f(n)) \to$

$n\to\infty \quad n^b \quad n\to\infty \quad bn^{b-1}$

$$= \lim_{n\to\infty} \frac{1}{b} \cdot n^{-b} = 0 \text{ if } b > 0.$$

If $a \neq 1$, $0^a = \lim_{n\to\infty} \left( \left( \frac{\log(n)}{n^{\frac{b}{a}}} \right) \right)^a$

zero

$$0 = \lim_{n\to\infty} \frac{\log(n)^a}{n^b}$$

$\square$

Ex. $2^n = o(4^n)$ $\quad \lim_{n\to\infty} \frac{2^n}{4^n} = \lim_{n\to\infty} \left(\frac{1}{2}\right)^n = 0$

Ex. $1 = \Theta(1{,}000{,}000)$

$\log(g(n)) = o(\log(f(n))) \to g(n) = o(f(n))$.

Ex. $f(n) = \log(n)^{\log(n)} \qquad g(n) = n^{100}$

$\log(f(n)) = \log^2(n) \qquad \log(g(n)) = 100 \log(n)$

$\log(g(n)) = o(\log(f(n))) \to g(n) = o(f(n))$.

$n\to\infty \quad \log(f(n)) \quad \log(\log(\log(n)))$

$\log(g(n)) = o(\log f(n)) \to$
$g(n) = o(f(n))$

Graphing calculators can sometimes lie.