September 25

Friday, September 25, 2020 3:33 PM

Comp Sci 212
1. Strong Induction
2. Invariants

Announcements
- Midterm October 19
- Office Hours update
- Getting to know you meetings
- Reading 9/28

Strong Induction
- Prove P(0) - Base case
- Prove if P(0), P(1), ..., P(n), then P(n+1)
  Inductive step

McDonald's sells chicken mcnuggets in packs of 3 and 7.
For what values of n can we buy exactly n chicken mcnuggets?

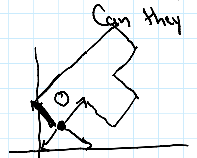
Theorem. For all integers n s.t. $n \geq 12$, you can buy exactly n chicken mcnuggets

$n=12$ - 3, 3, 3, 3   $n=15$ - 3, 3, 3, 3, 3
$n=13$ - 7, 3, 3   $n=16$ - 3, 7, 3, 3
$n=14$ - 7, 7   $n=17$ - 3, 7, 7

Proof. Use Induction.
Base case $n=12$ - 3, 3, 3, 3, $n=13$ - 7, 3, 3
$n=14$ - 7, 7
Inductive step - For $n \geq 15$. Assume you can buy exactly $n-3$ chicken mcnuggets. To buy n chicken mcnuggets, buy $n-3$ & pack of 3. □

If you can buy exactly 12, 13, 14, ..., $n-1$ ($n-3$) chicken mcnuggets, then you can buy n chicken mcnuggets.

Invariant - a property that stays the same over time.
Setting - bishop on a grid, starts at (1, 1).
Can only move diagonally.
Can they get to (1, 2)? (No)



(1, 1) → (0, 2) → (1, 3) → (2, 4) → (3, 3)
2   2   4   6   6

Thm. Let $(x_t, y_t)$ be position at time t. Then, $x_t + y_t$ is even. ← invariant
Proof. Use Induction.
Base Case - $x_1 = 1$, $y_1 = 1$, $x_1 + y_1 = 2$ even
Inductive step - Assume $x_t + y_t$ is even.
Then,
Case 1 - (move up, right) $x_{t+1} = x_t + 1$, $y_{t+1} = y_t + 1$
$x_{t+1} + y_{t+1} = x_t + y_t + 2$, still even
Case 2 - (move up, left)
$x_{t+1} = x_t - 1$ $y_{t+1} = y_t + 1$, $x_{t+1} + y_{t+1} = x_t + y_t$ still even.
Case 3 - (move down, right) $x_{t+1} = x_t + 1$
$y_{t+1} = y_t - 1$, $x_{t+1} + y_{t+1} = x_t + y_t$, still even
Case 4 - (move down, left) $x_{t+1} = x_t - 1$
$y_{t+1} = y_t - 1$, $x_{t+1} + y_{t+1} = x_t + y_t - 2$ still even □

Corollary. (1, 2) can not reached.
Proof. $1+2=3$ is odd.

General - Find a property / invariant
P(t) is true if the property holds at time t.
Show that P(t) is true for all t by induction.

In computer, loops
array / list A
for i from 1 to A.length
  do something
end

Invariant - after time i, subarray from 1 to i, has some property. (sorted). After the loop runs, entire array has property (sorted).

Puzzle

| | | |
|---|---|---|
| 1 | 2 | 3 |
| 4 | 5 | 6 |
| 7 | 8 | |

Figure 1
$n_1 = 0$

Is it possible to get to

| | | |
|---|---|---|
| 1 | 2 | 3 |
| 4 | 5 | 6 |
| 8 | 7 | |

Figure 2
$n_f = 1$

Invariant - # of pairs of square that are out of order ($n_t$ at time t)

| | | |
|---|---|---|
| 2 | 1 | 4 |
| 5 | 8 | 6 |
| 3 | | 7 |

(1, 2) (3, 4) (3, 5) (3, 8) (3, 6)
(6, 8) (7, 8)
7 pairs out of order

[illegible]

Inductive step - Assume $n_t$ is even. We will prove $n_{t+1}$ is even.
Row move - move within a row → keeps order the same
Column move - move within a column
→ Column move down
  Column move up

| | | |
|---|---|---|
| 1 | 2 | 3 |
| 4 | 5 | 6 |
| 7 | 8 | |

~~~ a, b, c ~~~
thing that's moved down
~~~ b, c, a, ~~~

Corollary. Figure 2 can not be reached from Figure 1.
Proof. Figure 2 has an odd # pairs out of order.
Figure 1 has an even # of pairs out of order.

| | | |
|---|---|---|
| 3 | | 7 |

7 pairs out of order

Theorem. $n_t$ is always even. (Invariant)

Use Induction.

Base case, $n_1 = 0$, no pairs are out of order.

thing that's moved down

~~~ b, c, a, ~~~

$(a, b) \longrightarrow (b, a)$

$(a, c) \longrightarrow (c, a)$

exactly one out of order

change is always +2, 0, or −2, therefore $n_t$ is always even.
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