September 23

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Comp Sci 212

1. Induction
2. Pitfalls of Induction
3. Strong Induction

Announcements

- Homework 1, out
- Advice, List of things you don't have to prove

Induction

- Prove $P(0)$ - Base Case
- Prove that for all n, if $P(n)$ then $P(n+1)$
  - Inductive step

Theorem. If $P(0)$ is true, and $P(n)$ implies $P(n+1)$ for all non-negative integers n, then $P(n)$ is true for all n (aka Induction works).

Proof. Use contradiction. Let n be the smallest non-negative integer for which $P(n)$ is false (axiom well-ordering principle). $n \neq 0$, because $P(0)$ is true. $P(n-1)$ is true, but $P(n-1)$ implies $P(n)$, which contradicts the fact that $P(n)$ is false.

(If $n=0$, $P(n-1) = P(-1)$)

Theorem. Let $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$, $n \geq 2$. (Fibonacci numbers)

$F_0 + F_1 + \dots + F_n = F_{n+2} - 1$ for all n.

$F_0 = 0$, $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, $F_4 = 3$, $F_5 = 5$, $F_6 = 8$

$n = 4$, $F_0 + F_1 + \dots + F_4 = 0+1+1+2+3 = 7 = F_6 - 1$

Proof. Use Induction.

Base case $n=0$, $0 = F_2 - 1 = 1 - 1 = 0$.

Inductive step: Assume $F_0 + \dots + F_n = F_{n+2} - 1$

Then $\underbrace{F_0 + \dots + F_n}$ $+ F_{n+1} = F_{n+2} - 1 + F_{n+1}$

$= F_{n+3} - 1$ (by def. of Fib numbers) □

Theorem. If $n \geq 4$, $2^n \leq 1 \cdot 2 \cdot 3 \cdots n$ $(n!)$

n	2^n	$n!$
1	2	1
2	4	2
3	8	6
4	16	24
5	32	120
6	64	720

Proof. Use Induction.

Base Case: If $n=4$, $2^n = 16 \leq n! = 24$.

Inductive step: Assume $n \geq 4$, $2^n \leq n!$. Additionally, $2 \leq n+1$. Therefore,

$2^n \cdot 2 \leq n!(n+1)$, i.e. $2^{n+1} \leq (n+1)!$ □

Pitfalls

Theorem. Every group of n people all have the same name

Proof. Use Induction.

Base case: $n=1$, obvious

Inductive step: Assume the statement is true for n. If there are $n+1$ people, number everyone from 1 to $n+1$.

$\underbrace{1 \; 2 \; 3 \; \cdots \; n}_{\text{all have the same name}} \; n+1$

People $1, \dots, n$ have the same name, and people $2, \dots, n+1$ have the same name (by the inductive hypothesis.) Therefore, $n+1$ has the same name as n, and people $1, \dots, n$, everyone has the same name. □

$P(1)$ does not imply $P(2)$

1 2 - groups don't overlap

$P(n)$ does imply $P(n+1)$ if $n \geq 2$

Strong Induction

- Prove $P(0)$ - Base case
- Prove if $P(0), P(1), .., P(n)$ are all true, then $P(n+1)$ is true.
  (Inductive step)

Theorem. Every non-negative integer at least 2 can be written as a product of primes (only divisible by 1 & itself).

Ex. $n=5$ $n=15=5\cdot3$

$n=7$ $n=24=3\cdot2\cdot2\cdot2$

Proof. Use strong induction.

Base case: $n=2$, 2 is prime

Inductive step: Assume thm is true up to & including $n-1$.

If n is prime, we're done.

If n is not prime, it's divisible by integer a s.t. $a\neq1$ and $a\neq n$

$n=a\cdot(\frac{n}{a})$, a and $\frac{n}{a}<n$, can be written as a product of primes, therefore so can n