

Comp Sci 212

1. Ift
2. Contradiction
3. Induction

Announcements

- TA/PM Office hours - starting 9/25
- Midterm/Final - two versions
→ December 7

Iff. $P \text{ iff } Q \rightarrow \text{If } P \text{ then } Q, \text{ and}$
 $\text{If } Q \text{ then } P$

Both implication & converse.

Theorem. For $a, b \geq 0$,

$$\frac{a+b}{2} = \sqrt{ab} \text{ iff } a=b.$$

Proof. If $a=b$, then $\frac{a+b}{2} = a = \sqrt{ab}$.

1st part $\rightarrow$ If $\frac{a+b}{2} = \sqrt{ab}$, then $a^2 + 2ab + b^2 = 4ab$,
2nd part $\rightarrow$ iff $a^2 - 2ab + b^2 = 0$, and $(a-b)^2 = 0$,
therefore $a-b=0$ and $a=b$.

iff $a-b=0$ and $a=b$.
iff $a=b$. $\square$

Combine iff, only need second part.

Def. x is even if $\frac{x}{2}$ is an integer.

Def. x is odd if $\frac{x-1}{2}$ is an integer.

Theorem Let x be an integer. Then x is even iff x^2 is even. $\rightarrow Q$

Proof. If x is even, $\frac{x}{2} = k$, where k is an integer. Therefore, $\frac{x^2}{4} = k^2$ and

iff $\frac{x^2}{4} = 2k^2$. $2k^2$ is an integer, so x^2 is even.

If x is odd, $\frac{x-1}{2} = k$ where k is an integer. Therefore, $\frac{x^2 - 2x + 1}{4} = k^2$ i.e.

$\frac{x^2 - 1}{4} = 2k^2 + x - 1$, and $2k^2 + x - 1$ is an integer, so x^2 is odd. $\square$

Contradiction. P is true $\rightarrow$ If not P then Q

If not P then not Q

Theorem. $\sqrt{2}$ is irrational - $\sqrt{2} \neq \frac{m}{n}$ for any pair of integers m or n .

Proof. Use contradiction. If $\sqrt{2}$ is rational, then $\sqrt{2} = \frac{m}{n}$ for integers m, n , assume in lowest terms. m, n can't both be even (otherwise divided both by 2).

$$2 = \frac{m^2}{n^2} \rightarrow 2n^2 = m^2 \rightarrow m^2 \text{ is even} \rightarrow m \text{ is even,}$$

$$m^2 \text{ is divisible by } 4 \rightarrow n^2 \text{ is even} \rightarrow n \text{ is even.}$$

$$\begin{aligned} m^2 &= 2n^2, \\ \frac{m^2}{4} &\text{ is an integer} \rightarrow \frac{n^2}{2} \text{ is an integer} \end{aligned}$$

$$\frac{m^2}{4} = \frac{n^2}{2}$$

Induction.

Theorem For all non-negative integers n ,

$$1+2+3+\dots+n = \frac{n(n+1)}{2}.$$

$$\begin{array}{ccc} 1+2+\dots+n & \frac{n(n+1)}{2} \\ n=1 & 1 & \frac{1(2)}{2} = 1 \end{array}$$

$$\begin{array}{ccc} n=2 & 1+2=3 & \frac{2(3)}{2} = 3 \end{array}$$

$$\begin{array}{ccc} n=3 & 1+2+3=6 & \frac{3(4)}{2} = 6 \end{array}$$

$$\begin{array}{ccc} n=4 & 1+2+3+4=10 & \frac{4(5)}{2} = 10 \end{array}$$

$$\begin{array}{ccc} n=5 & 1+2+3+4+5 & \frac{5(6)}{2} = 15 \end{array}$$

Strategy - Predicate $P(n)$ = "statement is true for n "

Prove $P(0) \rightarrow$ Base case

Prove for all n , if $P(n)$ then $P(n+1) \rightarrow$ Inductive Step
Then $P(n)$ is true for all n .

$$\begin{aligned} n=6 & 1+2+3+4+5+6 \\ \frac{5(6)}{2} + 6 &= \frac{5(6)+2(6)}{2} \\ &= \frac{6(7)}{2} \end{aligned}$$

Idea - Use solution for smaller n to prove the solution for larger n .

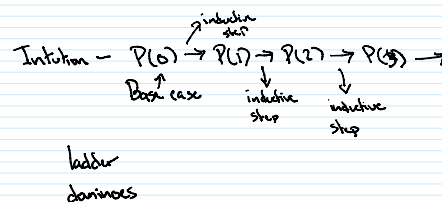
Theorem. For all non-negative integers n ,
 $1+2+3+\dots+n = \frac{n(n+1)}{2}$.

Proof. Use Induction.

Base Case $0 = \frac{0 \cdot 1}{2}$ ($1 = \frac{1 \cdot 2}{2}$ also works)

Inductive Step. If $1+2+3+\dots+n = \frac{n(n+1)}{2}$, then $1+2+3+\dots+n+(n+1) = \frac{n(n+1)}{2} + n+1 = \frac{n(n+1)+2(n+1)}{2} = \frac{(n+1)(n+2)}{2}$ $\square$

(If formula is true for n , then it's true for $n+1$)



$$n=5 \quad \frac{1+2+3+4+5}{2} = \frac{4(5)}{2} = \frac{4(5)+2(5)}{2} = \frac{4(5)}{2} \quad \text{solution for larger } n.$$

$$\frac{1+2+\dots+n-1}{2} + \frac{n}{2} = \frac{1+2+\dots+n}{2}$$

(If formula is true for n , then it's true for $n+1$) $\square$