September 21

Monday, September 21, 2020 3:33 PM

Comp Sci 212

1. Iff
2. Contradiction
3. Induction

Announcements

- TA / PM Office hours - starting 9/25
- Midterm / Final - two versions
  → December 7

Iff. P iff Q → If P then Q, and If Q then P

Both implication & converse.

Theorem. For $a, b \geq 0$,

$$\frac{a+b}{2} = \sqrt{ab} \text{ iff } a = b.$$

Proof. first part → If $a=b$, then $\frac{a+b}{2} = a = \sqrt{ab}$.

2nd part → If $\frac{a+b}{2} = \sqrt{ab}$, ~~then~~ iff $a^2+2ab+b^2 = 4ab$, ~~therefore~~ iff $a^2-2ab+b^2=0$, ~~and~~ iff $(a-b)^2=0$, ~~therefore~~ iff $a-b=0$ ~~and~~ iff $a=b$. □

Combine iff, only need second part.

Def. x is even if $\frac{x}{2}$ is an integer.

Def. x is odd if $\frac{x-1}{2}$ is an integer.

Theorem. Let x be an integer. Then x is even (P) iff x^2 is even. → Q

Proof. If P then Q: If x is even, $\frac{x}{2} = k$, where k is an integer. Therefore, $\frac{x^2}{4} = k^2$ and $\frac{x^2}{2} = 2k^2$. $2k^2$ is an integer, so x^2 is even.

If not P then not Q ↓ If Q then P: If x is odd, $\frac{x-1}{2} = k$ where k is an integer. Therefore, $\frac{x^2-2x+1}{4} = k^2$ i.e. $\frac{x^2-1}{2} = 2k^2+x-1$, and $2k^2+x-1$ is an integer, so x^2 is odd. □

Contradiction. P is true. → If not P then Q

If not P then not Q.

Theorem. $\sqrt{2}$ is irrational ~ $\sqrt{2} \neq \frac{m}{n}$ for any pair of integers m or n.

Proof. Use contradiction. If $\sqrt{2}$ is rational, then $\sqrt{2} = \frac{m}{n}$ for integers m & n, assume in lowest terms. m & n can't both be even (otherwise divide both by 2)

$2 = \frac{m^2}{n^2} \rightarrow 2n^2 = m^2 \rightarrow m^2$ is even $\rightarrow m$ is even,

m^2 is divisible by $4 \rightarrow n^2$ is even $\rightarrow n$ is even.

$m^2 = 2n^2$,

$\frac{m^2}{4}$ is an integer → $\frac{n^2}{2}$ is an integer

$\frac{m^2}{4} = \frac{n^2}{2}$

Induction.

Theorem. For all non-negative integers n, $1+2+3+\cdots+n = \frac{n(n+1)}{2}$.

	$1+2+\cdots+n$	$\frac{n(n+1)}{2}$
$n=1$	1	$\frac{(1)(2)}{2} = 1$
$n=2$	$1+2=3$	$\frac{2(3)}{2} = 3$
$n=3$	$1+2+3=6$	$\frac{(3)(4)}{2} = 6$
$n=4$	$1+2+3+4=10$	$\frac{4(5)}{2} = 10$
$n=5$	$1+2+3+4+5$	

$\frac{4(5)}{2} + 5 = \frac{4(5)+2(5)}{2} = \frac{6(5)}{2}$

→ $n=6$ $1+2+3+4+5+6$

$\frac{5(6)}{2} + 6 = \frac{5(6)+2(6)}{2} = \frac{6(7)}{2}$

Idea - Use solution for smaller n to prove the solution for larger n.

Strategy - Predicate $P(n)$ = "statement is true for n"

Prove $P(0)$ → Base case

Prove for all n, if $P(n)$ then $P(n+1)$. → Inductive Step

Then $P(n)$ is true for all n.

Intuition - $\underset{\uparrow \text{ Base case}}{P(0)} \xrightarrow{\text{induction step}} P(1) \xrightarrow{\text{inductive step}} P(2) \xrightarrow{\text{inductive step}} P(3) \rightarrow$

ladder

dominoes

Theorem. For all non-negative integers n, $1+2+3+\cdots+n = \frac{n(n+1)}{2}$.

Proof. Use Induction.

Base Case $0 = \frac{0 \cdot 1}{2}$ ($1 = \frac{1(2)}{2}$ also works)

Inductive Step. If $1+2+3+\cdots+n = \frac{n(n+1)}{2}$, then

$\underbrace{1+2+3+\cdots+n}+(n+1) = \frac{n(n+1)}{2} + n+1 = \frac{n(n+1)+2(n+1)}{2} = \underbrace{\frac{(n+2)(n+1)}{2}}_{\text{formula for } 1+2+3+\cdots+(n+1)}$ □

(If formula is true for n, then it's true for $n+1$)

$n=5$ $\quad \underbrace{1+2+3+4}+5$

$$\frac{4(5)}{2}+5=\frac{4(5)+2(5)}{2}=\frac{6(5)}{2}$$

solution for larger n.

$\square$

(If formula is true for n, then it's true for $n+1$)