November 23

Monday, November 23, 2020 4:06 PM

Comp Sci 212

1. Sensitivity Theorem

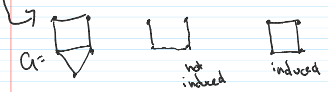
Announcements

- No lecture November 25
- No office hours November 27/29
- Final December 7

Def. Induced subgraph - $G=(V,E)$, $G'=(V',E')$ is an induced subgraph if $V' \subseteq V$, $\{v_1,v_2\} \in E$, $v_1,v_2 \in V'$ imply $\{v_1,v_2\} \in E'$.

Def. Hypercube - $G=(\{0,1\}^n, E)$

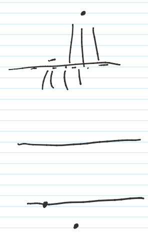
$E=\{\{v_1,v_2\} \mid v_1,v_2 \in \{0,1\}^n,\ v_1 \,\&\, v_2$ differ in 1 coordinate$\}$



Sensitivity Theorem - If $G=(\{0,1\}^n, E)$ is the hypercube, $G'=(V',E')$ is induced subgraph, $|V'| > \frac{2^n}{2}$ largest degree in G' is at least $\sqrt{n}$.

Hao Huang 2019.

$|V'| = \frac{2^n}{2}$



V_n = set of vertices w/ n 1's

V_{n-1} = w/ n-1 1's

$\vdots$

V_i = set of vertices w/ i 1's

$\vdots$

V_1

V_0 = set of vertices w/ 0 1's

Hypercube is bipartite

$|V_0 \cup V_2 \cup V_4 \cdots| = 2^n/2$

$|V_1 \cup V_3 \cup V_5 \cdots| = 2^n/2$

(binomial theorem)

$|V_i| = \binom{n}{i}$

If $V' = V_0 \cup V_2 \cup V_4 \cdots$

Induced subgraph has no edges, every vertex has degree 0.

Proof. A_n = a weighted adjacency matrix of hypercube (signed)

Let $A_0 = [0]$ $\qquad (A_n)_{i,j} = 1$ iff $\{i,j\} \in E$ in hypercube

$$A_n = \begin{bmatrix} A_{n-1} & I_{2^{n-1}} \\ I_{2^{n-1}} & -A_{n-1} \end{bmatrix}$$

$$A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad A_2 = \begin{array}{c|c} & \begin{matrix} 00 & 01 & 10 & 11 \end{matrix} \\ \begin{matrix} 00 \\ 01 \\ 10 \\ 11 \end{matrix} & \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix} \end{array}$$

$v_1 \,\&\, v_2$ be neighbors in hypercube

let i be the point where they differ

let d be the # of 1's before i, entry $(-1)^d$

$v_1 = 11$, $v_2 = 01$ $\quad d=0,\ (-1)^d = 1$

$v_1 = 11$, $v_2 = 10$ $\quad d=1,\ (-1)^d = -1$

Lemma. $A_n^2 = n \cdot I_{2^n}$

Proof. Use induction

Base case - $n=0$, $A_0^2 = 0 = 0 \cdot I_1$

Inductive step, assume $A_n^2 = n \cdot I_{2^n}$

Then, A_{n+1}^2

$$\begin{bmatrix} A_n & I_{2^n} \\ I_{2^n} & -A_n \end{bmatrix}\begin{bmatrix} A_n & I_{2^n} \\ I_{2^n} & -A_n \end{bmatrix} = \begin{bmatrix} A_n^2 + I_{2^n} & A_n - A_n \\ A_n - A_n & I_{2^n} + A_n^2 \end{bmatrix}$$

$$= \begin{bmatrix} n \cdot I_{2^n} + I_{2^n} & 0 \\ 0 & n \cdot I_{2^n} + I_{2^n} \end{bmatrix} = \begin{bmatrix} (n+1) I_{2^n} & \\ & (n+1) I_{2^n} \end{bmatrix} = (n+1) I_{2^{n+1}}$$

$\square$

$$B_n = \begin{bmatrix} A_{n-1} + \sqrt{n}\, I_{2^{n-1}} \\ I_{2^{n-1}} \end{bmatrix}$$

$2^n \times 2^{n-1}$ matrix $\rightsquigarrow$ columns of B_n are eigenvectors of A_n w/ eigenvalue $\sqrt{n}$

$A_n \cdot B_n$

$$\begin{bmatrix} A_{n-1} & I_{2^{n-1}} \\ I_{2^{n-1}} & -A_{n-1} \end{bmatrix}\begin{bmatrix} A_{n-1} + \sqrt{n}\, I_{2^{n-1}} \\ I_{2^{n-1}} \end{bmatrix} = \begin{bmatrix} A_{n-1}^2 + \sqrt{n} \cdot A_{n-1} + I_{2^{n-1}} \\ A_{n-1} + \sqrt{n}\, I_{2^{n-1}} - A_{n-1} \end{bmatrix}$$

$$= \begin{bmatrix} (n-1) I_{2^{n-1}} + I_{2^{n-1}} + \sqrt{n} \cdot A_{n-1} \\ \sqrt{n}\, I_{2^{n-1}} \end{bmatrix} = \begin{bmatrix} n I_{2^{n-1}} + \sqrt{n} \cdot A_{n-1} \\ \sqrt{n}\, I_{2^{n-1}} \end{bmatrix}$$

$= \sqrt{n} \cdot B_n$

Let x s.t. $(B_n x)_v = 0$ if $v \notin V'$

$B^* = B_n$, but only keep rows not in V'.

need $B^* x = 0$, B^* is $\underbrace{|V| - |V'|}_{\text{at most } 2^{n-1}-1} \times 2^{n-1}$

more unknowns than equations

Let $y = B_n x$

$A_n y = A_n B_n x = \sqrt{n} B_n x = \sqrt{n}\, y$

Let m be coordinate w/ max abs value in y.

$$\sqrt{n} \cdot |y|_m = |A_n y|_m = \Big|\sum_{v \in V} (A_n)_{mv} \cdot y_v\Big|$$

$$= \Big|\sum_{v \in V'} (A_n)_{mv} \cdot y_v\Big| \le \sum_{v \in V'} |(A_n)_{mv}| \cdot |y_v| \le |y_m| \cdot \sum_{\substack{v \in V' \\ v \text{ is neighbor of } m}} |\overbrace{(A_n)_{mv}}^{=0 \text{ if } m \,\&\, v \text{ are not neighbors}}| = |y_m| \cdot \underbrace{\deg(m)}_{\text{in induced subgraph.}}$$

$y_v = 0$ if $v \notin V'$