November 20

Friday, November 20, 2020 3:51 PM

Comp Sci 212

1. Boolean Hypercube

Announcements

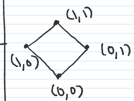
- Homework 8 due November 30
- No office hours November 27/29 (Thanksgiving)

Boolean hypercube $G = (\{0,1\}^n, E)$

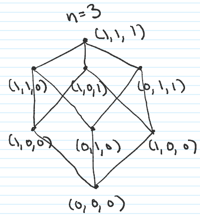
$E = \{\{v_1, v_2\} \mid v_1 + v_2 \text{ differ in 1 coordinate}\}$

2^n vertices $\quad \frac{n \cdot 2^n}{2}$ edges (degree each vertex is n)

Ex. $n=2$



$n=3$



1st — color 1

2nd — color 2

3rd

4th

Thm. Hypercube is bipartite

Prof. $f: \{0,1\}^n \to \{1,2\}$

$f(v) = \begin{cases} 1 & \text{if } v \text{ has an odd \# of 1's} \\ 2 & \text{if } v \text{ has even \# of 1's} \end{cases}$

If $\{v_1, v_2\}$ one of $v_1 + v_2$ must have an odd # of 1's, the other must have even # $\square$

Fact: every vertex has degree n.

Corollary: Adjacency matrix of G has $n + -n$ as eigenvalues.

$n = 0$, Adjacency matrix $= [0]$

1 vertex, no edges

A_n is adjacency matrix for hypercube on $\{0,1\}^n$

$$A_{n+1} = \begin{bmatrix} A_n & I_n \\ I_n & A_n \end{bmatrix}$$

$\underbrace{\quad}_{\frac{2^n}{2} \text{ vertices, start w/ 0}} \quad \underbrace{\quad}_{2^{n/2} \text{ vertices w/ 1}}$

Top left block → represent edges between vertices that start w/ 0

$(\cancel{0}, 1, 1, 0, 0)$

$(\cancel{0}, 1, 0, 0, 0)$

Top right block → represent edges between vertex starting w/ 0 $\quad (\cancel{0}, 1, 1, 0, 0)$

vertex starting w/ 1 $\quad (\cancel{1}, 1, 1, 0, 0)$

Let $V = \{0,1\}^n$ $\quad \mathbb{R}^V$ → think about coordinates in binary

2nd coordinate, $(0,0,1,0)^{th}$ coordinate

1 eigenvector per vertex

$f_v \in \mathbb{R}^V$ → eigenvector corresponding to vertex v.

$(f_v)_u = (-1)^{\langle v, u \rangle} \qquad \langle v, u \rangle = \sum_{i=1}^{n} v_i \cdot u_i$

$= \begin{cases} -1 & \text{if } \langle v, u \rangle \text{ is odd} \\ 1 & \text{if } \langle v, u \rangle \text{ is even} \end{cases}$

Ex. $v = (0, 0, 0, \dots 0)$. $\langle v, u \rangle = 0$ for all u

$(f_v)_u = 1$ for all u, f_v is all 1's vector.

$v = (1, 1, 1, 1) \quad \langle v, u \rangle = \#$ of 1's in u

$(f_v)_u = \begin{cases} 1 & \text{if } u \text{ has a even \# of 1's} \\ -1 & \text{if } u \text{ has an odd \# of 1's} \end{cases}$

same eigenvector as in bipartite

Thm. $Af_v = (n - 2 \cdot \# \text{ of 1's in } v) \cdot f_v$

$(n - 2 \cdot |v|) f_v$

$|v| = \#$ of 1's

Prof. $(Af_v)_w = \sum_{u \in \{0,1\}^n} A_{wu} \cdot (f_v)_u$

A_{wu}: $= 1$ if $\{w, u\}$ is edge ↳ only if $w + u$ differ in 1 coordinate

$= 0$ o.w.

$= \sum_{i=1}^{n} (f_v)_{w^{\oplus i}}$ $\quad w^{\oplus i}$ → w w/ i^{th} coordinate switched

$= \sum_{i=1}^{n} (-1)^{\langle v, w^{\oplus i} \rangle}$

$= \sum_{\substack{i \text{ s.t.} \\ v_i = 1}} (-1)^{\langle v, w \rangle} (-1) + \sum_{\substack{i \text{ s.t.} \\ v_i = 0}} (-1)^{\langle v, w \rangle}$

if $v_i = 1$, $\langle v, w \rangle = \langle v, w^{\oplus i} \rangle \pm 1$ $\qquad$ if $v_i = 0$, $\langle v, w \rangle = \langle v, w^{\oplus i} \rangle$

$= -|v| \cdot (-1)^{\langle v, w \rangle} + (n - |v|)(-1)^{\langle v, w \rangle}$

$= (n - 2|v|)(-1)^{\langle v, w \rangle}$

$= (n - 2|v|) \cdot (f_v)_w$ $\square$

Thm. If $u \neq v$, $\langle f_v, f_u \rangle = 0$.

Prof. $\sum_{w \in \{0,1\}^n} (f_v)_w \cdot (f_u)_w$

$= \sum_{w \in \{0,1\}^n} (-1)^{\langle v, w \rangle} (-1)^{\langle u, w \rangle}$

$= \sum_{w \in \{0,1\}^n} (-1)^{\cancel{\langle u, w \rangle + \langle v, w \rangle}} \langle u+v, w \rangle$

$\langle u, w \rangle = \sum_{i=1}^{n} u_i w_i, \quad \langle v, w \rangle = \sum_{i=1}^{n} v_i w_i$

$\langle u, w \rangle + \langle v, w \rangle = \sum_{i=1}^{n} u_i w_i + v_i w_i$

$= \sum_{i=1}^{n} (u_i + v_i) \cdot w_i = \langle u+v, w \rangle$

Enough to prove $\langle u+v, w \rangle$ is even for half of all w, odd for other half.

Let i be s.t. $u_i \neq v_i$

Then one of $\langle u+v, w \rangle$ and $\langle u+v, w^{\oplus i} \rangle$ is odd, the other is even.

$(u+v)_i = 1 \qquad \langle u+v, w \rangle = \langle u+v, w^{\oplus i} \rangle \pm 1$

essentially a bijection from set $\langle u+v, w \rangle$ is odd, and set $\langle u+v, w \rangle$ is even. $\square$