November 18

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Comp Sci 212

1. Spectral Graph Theory

Announcements
- Final - December 7, Crowdmark
- HW8 out, last homework

Def. Eigenvector v of matrix A

$Av = \lambda v$

λ: eigenvalue

Ex. $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 1 & -3 & -3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix} \begin{bmatrix} -2 \\ -2 \\ 0 \end{bmatrix} = -2 \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

What can linear algebra say about graphs?

Def. Adjacency matrix A of a graph $G=(V,E)$ matrix $A \in \mathbb{R}^{V \times V}$ s.t. $V=[n]$

$(A)_{i,j} = \begin{cases} 1 & \text{if } \{i,j\} \in E \\ 0 & \text{o.w.} \end{cases}$

Star



Adjacency matrix

$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & & & & & & \\ 1 & & & 0 & & & \\ \vdots & & & & & & \\ 1 & & & & & & \end{bmatrix}$

Cycle



$\begin{bmatrix} 0 & 1 & & & & & 1 \\ 1 & 0 & 1 & & & & \\ & 1 & \ddots & \ddots & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \ddots & \ddots & 1 & \\ & & & & 1 & 0 & \\ 1 & & & & & 1 & 0 \end{bmatrix}$

Complete graph $\begin{bmatrix} 0 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 0 \end{bmatrix}$

Thm. If every vertex in $G=(V,E)$ has degree d, $\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$ is an eigenvector w/ eigenvalue d.

Proof. Let A adjacency matrix of G,

$x = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$

$(Ax)_i = \sum_{v \in V} A_{i,v} \cdot x_v = \sum_{v \in V} A_{i,v} = \sum_{\substack{\text{Neighbors} \\ v \text{ of } i}} 1 = d$

$A_{i,v} = 1$ if v is a neighbor of i; 0 o.w. □

Thm. If $G=(V,E)$ is d-regular (every vertex has degree d) and has k connected components, there are k eigenvectors w/ eigenvalue d, orthogonal to each other. ($x^{(1)}, \ldots, x^{(k)}$, $\langle x^{(i)}, x^{(j)} \rangle = 0$ for all $i \neq j$.)

Proof. Let $V_1, \ldots, V_k$ be the sets of vertices each connected component.

$\begin{array}{c} \\ V_1 \\ V_2 \\ V_3 \\ \vdots \end{array} \begin{array}{c} \begin{array}{ccc} V_1 & V_2 & V_3 \cdots \end{array} \\ \begin{bmatrix} \square & & 0 \\ & \square & \\ 0 & & \square \end{bmatrix} \end{array}$

$x^{(i)} \in \mathbb{R}^V \quad (x^{(i)})_v = \begin{cases} 1 & \text{if } v \in V_i \\ 0 & \text{o.w.} \end{cases}$

By previous thm, $x^{(i)}$ has eigenvalue d.

Because V_i, V_j are disjoint if $i \neq j$, $\langle x^{(i)}, x^{(j)} \rangle = 0$.

Thm. If $G=(V,E)$ is d-regular and bi-partite there is 1 eigenvector w/ eigenvalue $-d$.

Proof. Let L and R be partition of V s.t. every edge goes from L to R.

Let $x \in \mathbb{R}^V$, $x_v = \begin{cases} 1 & \text{if } v \in L \\ -1 & \text{if } v \in R \end{cases}$

$\begin{array}{c} \\ L \\ R \end{array} \begin{array}{c} \begin{array}{cc} L & R \end{array} \\ \begin{bmatrix} 0 & \square \\ \square & 0 \end{bmatrix} \end{array}$

Let A be adjacency matrix

If $i \in L$

$(Ax)_i = \sum_{v \in V} A_{i,v} x_v = -\sum_{v \in R} A_{i,v}$ (if $v \in L$, $A_{i,v} = 0$)

$= -\sum_{\substack{\text{Neighbors} \\ i \text{ of } v}} 1 = -d$

Similarly if $i \in R$, $(Ax)_i = d$. □

Thm. If $G=(V,E)$ is d-regular, all eigenvalues are at most d in absolute value of adjacency matrix

Proof. Let x be any eigenvector of adjacency matrix A.

Let m be the coordinate of x w/ max absolute value.

$|(Ax)_m| = |\lambda x_m|$ for some λ-eigenvalue

$= \left| \sum_{v \in V} A_{mv} \cdot x_v \right| = \left| \sum_{\substack{\text{Neighbors } v \\ \text{of } m}} A_{mv} \cdot x_v \right|$

$\leq \sum_{\substack{\text{Neighbors } v \\ \text{of } m}} |A_{mv} \cdot x_v| = \sum_{\substack{\text{Neighbors } v \\ \text{of } m}} |x_v|$

$\leq \sum_{v \in N(m)} |x_m| = d|x_m|$. → $|\lambda||x_m| \leq d|x_m|$ → $|\lambda| \leq d$

Def. A matrix A is symmetric if $A_{ij} = A_{ji}$ for all i, j.

Thm Spectral theorem - If $A \in \mathbb{R}^{n \times n}$ is symmetric, there exist n eigenvectors, are mutually orthogonal, real eigenvalues.

Fact - Adjacency matrix of undirected graph, is symmetric.