November 11

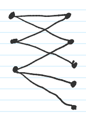
Wednesday, November 11, 2020 3:54 PM

Comp Sci 212

1. Bipartite graphs

2. Matchings

Def. If $G=(V,E)$ is 2-colorable, G is bipartite.



Thm. $G=(V,E)$ is bipartite iff it has no odd-length cycles.

Proof. If G has an odd cycle it is not 2-colorable (last class).

If G has no odd length cycles

Def. length of a path is number of edges.

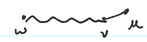
Def. $dist(u,v)$ = length shortest path from u to v.

Pick arbitrary $w \in V$ $\quad f: V \to [2]$

$f(v) = \begin{cases} 1 & \text{if } dist(w,v) \text{ is odd} \\ 2 & \text{if } dist(w,v) \text{ is even.} \end{cases}$

Need to prove f is a valid 2-coloring.

Use contradiction, assume $\{v,u\} \in E$ s.t. $f(v)=f(u)$



$dist(w,u) = dist(w,v)-1$

$dist(w,v)$

$dist(w,v)+1$

Because $f(u)=f(v)$, $dist(w,v)=dist(w,u)$.

Let $(w, v_1, v_2, \ldots, v_\ell, v)$ path w to v

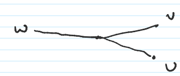
$(w, u_1, u_2, \ldots, u_\ell, u)$ path w to u.

If $v_i \neq u_i$ for all i,

$(v, v_\ell, \ldots, v_1, w, u_1, u_2, \ldots, u_\ell, u, v)$ $\to$ $dist(w,v_i) = dist(w,u_i) = i$

odd-length cycle (length $= 2\ell+3$)

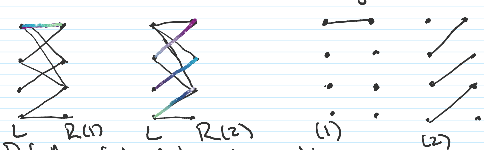
If $v_i = u_j$, then $i=j$.



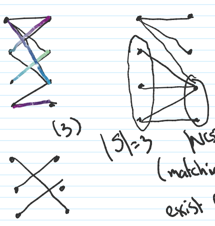
Let i be the largest number s.t. $u_i = v_i$

$(v, v_\ell, \ldots, v_i, u_{i+1}, u_{i+2}, \ldots, u_\ell, v)$ - odd length

Def. Matching is a subset of edges s.t. each vertex is in at most one edge.



Def. A perfect matching is a matching where each vertex is in exactly 1 edge. (sometimes, only left vertices)



- no perfect matching exists

$|S|=3 \quad |N(S)|=2$

(matchings / perfect matchings exist for non-bipartite graph)

What conditions guarantee a perfect matching?

$G=(V,E)$ bi-partite, $V = L \cup R$

L set of vertices on left side

R right side

$N(S) = \{v \mid \{u,v\} \in E, u \in S\}$ $\quad S \subseteq V$

= set of neighbors of S

The matching condition - If $S \subseteq L$, then $|N(S)| \geq |S|$ (needs to be true for every $S \subseteq L$)

Thm. A bipartite graph satisfies the matching condition iff there exists a perfect matching.

$\hookrightarrow$ match vertices on left-hand

Proof. If G has a perfect matching,

Let M be that matching,

$M(S)$ = set of vertices matched to S.

$|S| = |M(S)|$. $M(S) \subseteq N(S)$, $|M(S)| \leq |N(S)|$,

$\hookrightarrow$ perfect matching

so $|S| \leq |N(S)|$.

Other direction - induction on $|L|$

Base case - $|L|=1$, match $v \in L$ w/ anything

Inductive step: Assume thm is true if $|L| \leq k$, assume $|L| = k+1$.

Either $|N(S)| = |S|$ for some S s.t. $|S| \leq k$, - Case 2

or $|N(S)| > |S|$ for every S s.t. $|S| \leq k$ - Case 1

($S \subseteq L$ for both cases)

Case 1 - Pick $v \in L$, match v w/ any neighbor. Remove v & its match

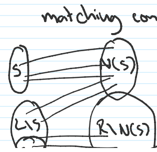
Matching condition still holds on new graph, use solution from inductive hypothesis.

Case 2 - Pick an S st. $|N(S)| = |S|$.

Match S with $N(S)$ $\Big\}$ inductive hypothesis

Match $L \setminus S$ with $R \setminus N(S)$

$\hookrightarrow$ need to prove this satisfies matching condition, can obtain perfect matching for entire graph.



$\to N_0(S_0)$

Assume you can't match $L \setminus S$ w/ $R \setminus N(S)$. (contradiction). By inductive hypothesis, there exist $S_0 \subset L \setminus S$ s.t.

$|\{v \mid \{u,v\} \in E, u \in S_0, v \in R \setminus N(S)\}| < |S_0|$

$N_0(S_0)$ = set of neighbors of S_0 in $R \setminus N(S)$.

$|N_0(S_0)| < |S_0|$.

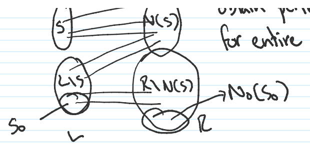
$S \cup S_0 \qquad N(S \cup S_0) = N(S) \cup N_0(S_0)$

$|N(S \cup S_0)| = |N(S)| + |N_0(S_0)|$

sum rule

$< |S| + |S_0|$

$= |S \cup S_0|$.



obtain perfect matching
for entire graph.

Sum rule
$< |S| + |S_0|$
$= |S \cup S_0|$.

contradicts assumption that m.c. is satisfied in original graph