Recall: Matching Algorithms

Alg: Ascending Prices (AP)

0. initialize prices to $p = 0$.
1. Construct demand graph D
2. if D has perfect matching output it (i.e. if p is market clearing)
3. else:
   a) find S, minimum violation of Hall's Thm
   b) increase prices of $N(S)$ until demand set of buyers $i \in S$ changes
   c) repeat (1).

Analysis

Thm: AP maximizes welfare

Proof: primal = dual



"For any maximization problem, corresponding minimization problem with the same optimal value"

Primal Program

$x_{ij} = \begin{cases} 1 & \text{if assign } j \text{ to } i \\ 0 & \text{o.w.} \end{cases}$

$$\text{Primal}(x) = \max_{x} \sum_{ij} v_{ij} x_{ij}$$

$$\text{s.t.} \quad \sum_i x_{ij} \le 1 \quad \forall j$$

$$\sum_j x_{ij} \le 1 \quad \forall i$$

$$x_{ij} \ge 0 \quad \forall i,j$$

Dual Program

$$\text{Dual}(u, p) = \min_{u,p} \sum_i u_i + \sum_j p_j$$

$$\text{s.t.} \quad u_i + p_j \ge v_{ij} \quad \forall i,j$$

$$u_i \ge 0 \quad \forall i$$

$$p_j \ge 0 \quad \forall j$$

Intuition:

- utilities u_i, prices p
- $u_i \ge v_{ij} - p_j$

x^*, u^*, p^* with $\text{Primal}(x^*) = \text{Dual}(u^*, p^*)$

Lemma 1: $\text{Primal}(x) \le \text{Dual}(u, p)$ for all feasible x & u, p

Lemma 2: Alg's termination conditions identify x^*, u^*, p^*

Proof of 1:

- any feasible x, any feasible (u, p)

$$\text{Primal}(x) = \sum_{ij} v_{ij} x_{ij}$$

[dual feasibility] $$\leq \sum_{ij} (u_i + p_j) x_{ij}$$

$$= \sum_i \sum_j u_i x_{ij} + \sum_j \sum_i p_j x_{ij}$$

$$= \sum_i u_i \sum_j x_{ij} + \sum_j p_j \sum_i x_{ij}$$

[primal feasibility] $$\leq \sum_i u_i \cdot 1 + \sum_j p_j \cdot 1$$

$$= \text{Dual}(u, p)$$

Proof of 2:

- x^* = matching when alg terminates.
- p^* = (market clearing) prices p^*
- u^* = utilities from p^* (for highest utility item under p^*)
- perfect matching of demand sets
  - for i, j with $x^*_{ij} = 1$: $u_i = v_{ij} - p_j$ $\Rightarrow$ equality
  - for all i, j: $u_i \geq v_{ij} - p_j \Rightarrow$ feasible dual
- x^* is perfect match so primal feasibility
  inequality is equality.

Primal Program

$$x_{ij} = \begin{cases} 1 & \text{if assign } j \text{ to } i \\ 0 & \text{o.w.} \end{cases}$$

$$\text{Primal}(x) = \max_x \sum_{ij} v_{ij} x_{ij}$$

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Intuition:

- utilities u, prices p $\Rightarrow$ x^*, u^*, p^* with $\text{Primal}(x^*) = \text{Dual}(u^*, p^*)$
- $u_i \geq v_{ij} - p_j$

Lemma 1: $\text{Primal}(x) \leq \text{Dual}(u, p)$ for all feasible x & u, p

Lemma 2: Alg's termination conditions identify x^*

Online Matching

"match online items to offline buyers"

Setup

- n buyers, n items
- buyer i has value v_i for items $S_i \subseteq \{1, \ldots, n\}$
- initially all buyers present.
- in round j:
  - item j arrives.
  - match j to any remaining buyer i with $j \in S_i$
  - matched buyer leaves the market.

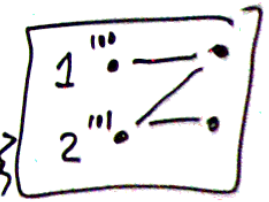
Goal: maximize welfare.

Alg: Greedy Online Matching

- in round j:
  match j to remaining buyer i with highest value for j.

Example

- $v_1 = 100$; $S_1 = \{A\}$
- $v_2 = 101$; $S_2 = \{A, B\}$



- Greedy:
  - A arrives, matched to 2.
  - B arrives, not matched.
- Greedy: 101
- OPT: 201

$\Rightarrow$ 2-approximation.

Q: can it be worse? A: no!

Thm: Greedy Online Matching is 2-approx.

Proof Idea:

(a) each edge in greedy blocks at most two edges of OPT.

(b) blocked edges are lower value.

$\Rightarrow 2 \cdot \text{Greedy} \geq \text{OPT}$.

Example

OPT
- $(1, A) = 100$
- $(2, B) = 101$

Greedy:
$(2, A) = 101$

- Greedy chose $(2, A)$ over $(1, A)$ because $101 = v_2 > v_1 = 100$.
- Greedy sets same value for $(2, A)$ as $(2, B)$.

Online Matching Mechanism

Note: allocation rule of Greedy is monotonic.

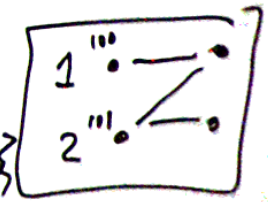
Mech: Online greedy threshold pricing
- run greedy
- charge winners "minimum bid that wins"?

Thm: truthtelling is PSE.

Cor: 2-approx in PSE.

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