CS 396: Online Markets

Lecture 17: Online Matching

Last Time:

- matching markets
- maximum weight matching
- market clearing
- externality pricing mechanism
  (a.k.a, Vickrey-Clarke-Groves, VCG)

Today:

- maximum weight matching (cont)
- duality
- online matching

Exercise: Externality Pricing

Recall:

- externality pricing mechananism:
  - pick the outcome that maximizes the total welfare.
  - charge each buyer the difference between the optimal welfare without the buyer and the welfare of other buyers (in the optimal welfare outcome)

Setup:

- two buyers 1 and 2, two houses A and B
- bids:

	House A	House B
Buyer 1	8	7
Buyer 2	6	3

Questions:

- Which house does Buyer 2 get in the externality pricing mechanism?
- What is Buyer 2's payment?

Matching Algorithms

Alg: Ascending Prices (AP)

0. initialize prices: $\mathbf{p} = \mathbf{0}$
1. Construct demand graph D
2. if D has perfect matching, output it and halt. (i.e., if $\mathbf{p}$ are market clearing)
3. else,
   a) find set S "minimally" violating Hall's Thm
   b) increase prices of $N(S)$ until demand set of buyer $i \in S$ changes.
   c) repeat (1)

Offline Matching Mechanisms

Recall: Externality Pricing Mechanisms maximizes welfare in dominant strategy equilibrium

Mech: Ascending Auction (AA)

"implement ascending prices (AP) as auction"

Thm: EP's prices = AP's prices.

Cor: "truthtelling" is dominant strategy in AA.

Analysis of Ascending Prices Algorithm

Thm: Ascending Prices Alg maximizes welfare

Proof: Primal = Dual

"for maximization problem, corresponding minimization problem"

Primal Program:

$$\begin{aligned} \text{Primal}(\mathsf{x}) = \max_{\mathsf{x}} & \sum\nolimits_{ij} \mathsf{v}_{ij}\, \mathsf{x}_{ij} & \\ \text{s.t.} & \sum\nolimits_{j} \mathsf{x}_{ij} \le 1 & \forall i \\ & \sum\nolimits_{i} \mathsf{x}_{ij} \le 1 & \forall j \\ & \mathsf{x}_{ij} \ge 0 & \forall i, j \end{aligned}$$

Dual Program:

$$\begin{aligned}\text{Dual}(\mathbf{u},\mathbf{p}) = \min_{\mathbf{u},\mathbf{p}} & \sum\nolimits_i \mathsf{u}_i + \sum\nolimits_j \mathsf{p}_j \\ \text{s.t. } & \mathsf{u}_i + \mathsf{p}_j \geq \mathsf{v}_{ij} && \forall i,j \\ & \mathsf{u}_i \geq 0 && \forall i \\ & \mathsf{p}_j \geq 0 && \forall j\end{aligned}$$

Intuition:

- utilities $\mathbf{u}$ and prices $\mathbf{p}$
- $\mathsf{u}_i \geq \mathsf{v}_{ij} - \mathsf{p}_i$

Lemma 1: $\text{Primal}(\mathbf{x}) \leq \text{Dual}(\mathbf{u},\mathbf{p})$

Lemma 2: alg's termination condition identifies dual solution with value equal to primal.

Proof 1:

- any primal feasible $\mathbf{x}$
- any dual feasible $\mathbf{u},\mathbf{p}$

$$\begin{aligned}\text{Primal}(\mathbf{x}) &= \sum\nolimits_{ij} \mathsf{v}_{ij}\,\mathsf{x}_{ij} \\ \text{[dual feasibility]} \quad &\leq \sum\nolimits_{ij} (\mathsf{u}_i + \mathsf{p}_j)\,\mathsf{x}_{ij} \\ &= \sum\nolimits_i \sum\nolimits_j \mathsf{u}_i\,\mathsf{x}_{ij} + \sum\nolimits_j \sum\nolimits_i \mathsf{p}_j\,\mathsf{x}_{ij} \\ &= \sum\nolimits_i \mathsf{u}_i \sum\nolimits_j \mathsf{x}_{ij} + \sum\nolimits_j \mathsf{p}_j \sum\nolimits_i \mathsf{x}_{ij} \\ \text{[primal feasibility]} \quad &\leq \sum\nolimits_i \mathsf{u}_i + \sum\nolimits_j \mathsf{p}_j = \text{Dual}(\mathbf{u},\mathbf{p})\end{aligned}$$

Proof 2:

- for prices $\mathbf{p}$ and allocation $\mathbf{x}$ from algorithm,
- set $\mathbf{u}$ as utilities of buyers
- $\mathsf{u}_i = \mathsf{v}_{ij} - \mathsf{p}_j$ if $\mathsf{x}_{ij} = 1$
- perfect matching of demand sets
  $\Rightarrow \forall i,j:\ \mathsf{u}_i \geq \mathsf{v}_{ij} - \mathsf{p}_j$
  $\Rightarrow$ dual feasibility
- inequalities are equalities in proof of Lemma 1.
  $\Rightarrow$ primal = dual.

Exercise: Matching Dual

Recall:

$$\begin{aligned}\text{Dual}(\mathbf{u},\mathbf{p}) = \min_{\mathbf{u},\mathbf{p}} & \sum\nolimits_i \mathsf{u}_i + \sum\nolimits_j \mathsf{p}_j \\ \text{s.t. } & \mathsf{u}_i + \mathsf{p}_j \geq \mathsf{v}_{ij} && \forall i,j \\ & \mathsf{u}_i \geq 0 && \forall i \\ & \mathsf{p}_j \geq 0 && \forall j\end{aligned}$$

Setup:

- two buyers 1 and 2, two houses A and B
- values:

	House A	House B
Buyer 1	8	7
Buyer 2	6	3

Questions: Identify dual utilities:

- u_1?
- u_2?

Online Matching

"match offline buyers to online items"

Setup:

- n buyers, n items.
- buyer has value v_i for any item in $S_i \subset \{1, \ldots, n\}$
- initially all buyers present
- in round j,
  - item j arrives.
  - match to any remaining buyer i with $S_i \ni j$.
  - matched buyer leaves

Goal: maximize welfare = sum of values of matched buyers.

Alg: Greedy Online Matching

- in round j:
  - match to remaining buyer with highest value

Q: is this algorithm good?

Example:

- $\mathsf{v}_1 = 100$; $S_1 = \{A\}$
- $\mathsf{v}_2 = 101$; $S_2 = \{A, B\}$
- Greedy:
  - A arrives, assigned to 2.
  - B arrives, not assigned.
- Greedy = 101
- OPT = 201

  ⇒2 approximation

Q: can it be worse?

A: no.

Thm: Greedy Online Matching is a 2-approximation

Proof: - Approach: (a) each edge of Greedy blocks at most two edges of OPT (b) these blocked edges are lower value

- consider (i, j) matched by OPT
- suppose (i, j) matched by Greedy:
  - charge v_i to (i, j)
- suppose (i, j) not matched by Greedy
- at time j:
  - if i is already matched to j':
    - charge v_i to (i, j') (i has the same value for j and j').
  - else, j is already matched to i':
    - when j arrived
    - greedy choose i' instead of i (so $\mathsf{v}_{i'} \geq \mathsf{v}_i$)
    - charge v_i to (i', j)
- each edge in Greedy charged at most twice
- all edges in OPT are accounted for ⇒ 2Greedy ≥ OPT.

Example:

- $\mathsf{v}_1 = 100$; $S_1 = \{A\}$
- $\mathsf{v}_2 = 101$; $S_2 = \{A, B\}$
- $(1, A)$ not matched by greedy.
  - charged to $(2, A)$ with $\mathsf{v}_2 > \mathsf{v}_1 = 100$
- $(2, B)$ is not matched by greedy.
  - charged to $(2, A)$ with value $\mathsf{v}_2 = \mathsf{v}_2 = 101$
- 2Greedy $= 202 \geq 100 + 101 =$ OPT.

Online Matching Mechanisms

Note: allocation rule of Greedy Online Matching is monotonic

Mech: Online Greedy Threshold Pricing

- run Online Greedy Algorithm.
- charge buyer minimum bid needed to win.

Thm: Truthtelling is DSE in Online Greedy Threshold Pricing Mechanism

Cor: Online Greedy Threshold is 2-approx in DSE.