CS 396: Online Markets

Lecture 16: Offline Matching

Last Time:

- value inference (econometrics)
- inference for learning bidders

Today:

- externality pricing mechanism (a.k.a, Vickrey-Clarke-Groves, VCG)
- matching markets
- maximum weight matching
- market clearing
- duality

Exercise: House Allocation

Setup:

- two buyers 1 and 2
- two houses A and B
- values:

	House A	House B
Buyer 1	8	7
Buyer 2	6	3

Questions:

- What house does buyer 1 get in the welfare maximizing matching?
- What is the welfare of the optimal matching?

Matching Markets

E.g.

- eBay: sellers and buyers
- ad auctions: advertisers and users
- uber: drivers and riders

Typically:

- one side is long-lived and strategic
- one side is short-lived and behavioral

Setup:

- n buyers (strategic), n items (non-strategic)
- buyers want an item
- items can be sold to a buyer
- buyer i's value for item j: v_{ij}
- goal: maximize welfare

A.k.a.: maximum weighted bipartite maching

Market Clearing

"prices where there is no contention for items, and unsold items have price 0"

Recall: (unweighted) bipartite graphs (A, B, E)

PICTURE

Recall: perfect matching

Recall: Hall's Theorem: a bipartite graph (A, B, E) has perfect matching iff all $S \subset A$ has $|S| \leq |N(S)|$

Def: (bipartite) **demand graph** D at prices $\mathbf{p}$ is: $N(i) = \operatorname{argmax}_j \mathsf{v}_{ij} - \mathsf{p}_j$

Def: prices $\mathbf{p}$ are **market clearing** if demand graph has perfect matching

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Matching Algorithms

Alg: Ascending Prices (AP)

0. initialize prices: $\mathbf{p} = \mathbf{0}$
1. Construct demand graph D
2. if D has perfect matching, output it and halt. (i.e., if $\mathbf{p}$ are market clearing)
3. else,
   a) find set S "minimally" violating Hall's Thm
   b) increase prices of $N(S)$ until demand set of buyer $i \in S$ changes.
   c) repeat (1)

Example:

	A	B	C
1	9	8	0
2	7	6	2
3	6	2	4

Exercise: House Pricing

Setup:

- two buyers 1 and 2
- two houses A and B
- values:

	House A	House B
Buyer 1	8	7
Buyer 2	6	3

Questions:

- Are prices $\mathsf{p}_A = 5$ and $\mathsf{p}_B = 3$ market clearing?
- Are prices $\mathsf{p}_A = 7$ and $\mathsf{p}_B = 7$ market clearing?
- What is price for House A in Ascending Prices Algorithm?
- What is price for House B in Ascending Prices Algorithm?

Thm: Ascending Prices Alg maximizes welfare

Proof: Primal = Dual

"for maximization problem, corresponding minimization problem"

Primal Program:

$$\begin{aligned} \text{Primal}(\mathbf{x}) = \max_{\mathbf{x}} & \sum\nolimits_{ij} \mathsf{v}_{ij}\, \mathsf{x}_{ij} & \\ \text{s.t.} & \sum\nolimits_{j} \mathsf{x}_{ij} \leq 1 & \forall i \\ & \sum\nolimits_{i} \mathsf{x}_{ij} \leq 1 & \forall j \\ & \mathsf{x}_{ij} \geq 0 & \forall i, j \end{aligned}$$

Dual Program:

$$\begin{aligned} \text{Dual}(\mathbf{u}, \mathbf{p}) = \min_{\mathbf{u},\mathbf{p}} & \sum\nolimits_i \mathsf{u}_i + \sum\nolimits_j \mathsf{p}_j & \\ \text{s.t.}\ & \mathsf{u}_i + \mathsf{p}_j \geq \mathsf{v}_{ij} & \forall i, j \\ & \mathsf{u}_i \geq 0 & \forall i \\ & \mathsf{p}_j \geq 0 & \forall j \end{aligned}$$

Intuition:

- utilities $\mathbf{u}$ and prices $\mathbf{p}$
- $\mathsf{u}_i \geq \mathsf{v}_{ij} - \mathsf{p}_i$

Lemma 1: $\text{Primal}(\mathbf{x}) \leq \text{Dual}(\mathbf{u}, \mathbf{p})$

Lemma 2: alg's termination condition identifies dual solution with value equal to primal.

Proof 1:

- any primal feasible $\mathbf{x}$
- any dual feasible $\mathbf{u}, \mathbf{p}$

$$\begin{aligned} & \text{Primal}(\mathbf{x}) = \sum\nolimits_{ij} \mathsf{v}_{ij}\, \mathsf{x}_{ij} \\ \text{[dual feasibility]} \quad & \leq \sum\nolimits_{ij} (\mathsf{u}_i + \mathsf{p}_j)\, \mathsf{x}_{ij} \\ & = \sum\nolimits_i \sum\nolimits_j \mathsf{u}_i\, \mathsf{x}_{ij} + \sum\nolimits_j \sum\nolimits_i \mathsf{p}_j\, \mathsf{x}_{ij} \\ & = \sum\nolimits_i \mathsf{u}_i \sum\nolimits_j \mathsf{x}_{ij} + \sum\nolimits_j \mathsf{p}_j \sum\nolimits_i \mathsf{x}_{ij} \\ \text{[primal feasibility]} \quad & \leq \sum\nolimits_i \mathsf{u}_i + \sum\nolimits_j \mathsf{p}_j = \text{Dual}(\mathbf{u}, \mathbf{p}) \end{aligned}$$

Proof 2:

- for prices $\mathbf{p}$ and allocation $\mathbf{x}$ from algorithm,
- set $\mathbf{u}$ as utilities of buyers
- $\mathsf{u}_i = \mathsf{v}_{ij} - \mathsf{p}_j$ if $\mathsf{x}_{ij} = 1$
- perfect matching of demand sets
  $\Rightarrow \forall i, j:\ \mathsf{u}_i \geq \mathsf{v}_{ij} - \mathsf{p}_j$
  $\Rightarrow$ dual feasibility
- inequalities are equalities in proof of Lemma 1.
  $\Rightarrow$ primal = dual.

Offline Matching Mechanisms

Mech: Externality Pricing (EP)

0. solicit bids.
1. Compute optimal welfare $W = \text{OPT}(\mathbf{b})$ and outcome $\mathbf{x}$
2. Compute optimal welfare without bidder i: $W_i = \text{OPT}(\mathbf{b}_{-i})$
3. Charge bidders **externality**: $\mathsf{p}_i = W_i - (W - \mathsf{b}_{ij}\mathsf{x}_{ij})$

A.k.a.: Vickrey-Clarke-Groves (VCG) Mechanism

Thm: Externality Pricing Mechanism is truthful.

Proof:

- consider alternative payment $\mathsf{p}'_i = -(W - \mathsf{b}_{ij}\mathsf{x}_{ij})$

  "pay bidder value of others"

- truthtelling utility is $\mathsf{v}_{ij}\mathsf{x}_{ij}) + (W - \mathsf{b}_{ij}\mathsf{x}_{ij}) = W$

  "bidder's utility equals societies welfare"

- EP maximizes societies welfare on truthful bids

  $\Rightarrow$ optimal to bid truthfully.

- p'_i is the same as p_i except for "constant" W_i

  "constants don't affect strategies"

Offline Matching Mechanisms (Revisited)

Mech: Ascending Auction (AA)

"implement ascending prices (AP) as auction"

Q: What are good strategies?

A: "report demand sets truthfully"

Thm: EP's prices = AP's prices.

Cor: "truthtelling" is dominant strategy in AA.
