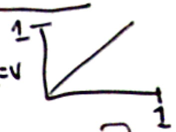
Recall: Auction Revenue

Def: allocation rule

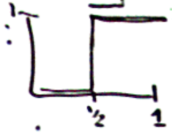
$x(v) = \Pr[\text{bidder wins with value } v]$

Ex B.1

A: $x_A(v) = v$



B: $x_B(v)$:



Recall:

- welfare is value of winner
- expected welfare is $E_{v\sim F}[v\, x(v)]$

Thm: expected revenue is

$$E_{v\sim F}\Big[\underbrace{\Big[v - \frac{1-F(v)}{f(v)}\Big]}_{\phi(v)} x(v)\Big]$$

Def

- virtual value: $\phi(v) = v - \frac{1-F(v)}{f(v)}$
- expected virtual welfare: $E_{v\sim F}[\phi(v)\cdot x(v)]$

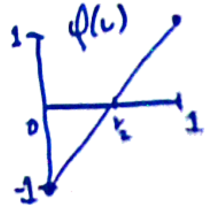
Conclusion: expected revenue = expected virtual surplus.

Q: how to maximize virtual welfare?

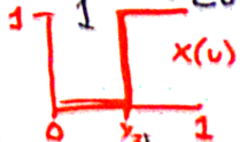
A: allocate to v if $\phi(v) \geq 0$

$\Rightarrow \phi^{-1}(\phi(v)) \geq \phi^{-1}(0) \Rightarrow v \geq \phi^{-1}(0)$.

Example:



- $v \sim U[0,1]$
- $F(v) = v$; $f(v) = 1$
- $\phi(v) = v - \frac{1-F(v)}{f(v)} = v - \frac{1-v}{1} = 2v-1$.
- $\phi^{-1}(0) = ½$
- optimal to allocate if $v \geq ½$.
- posting price $\hat{v} = ½$.



Cor: optimal single buyer mechanism posts price $\hat{v} = \phi^{-1}(0)$.

Q: what is expected virtual surplus $E[\phi(v)\, x(v)]$

$= \int_0^1 \phi(v)\, x(v)\, dv = ¼$

Q: what is expected revenue from posting price ½? ¼

Revelation Principle

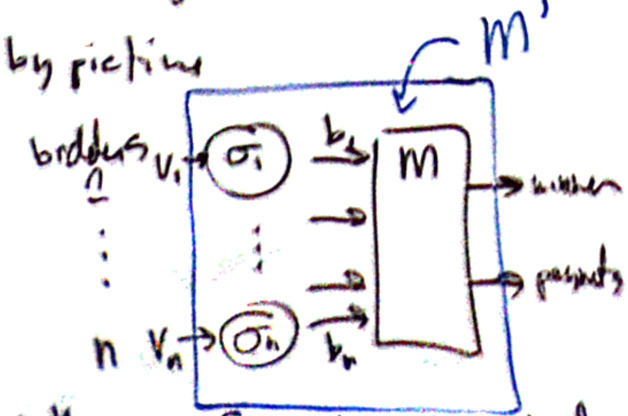
"may as well only consider mechanisms with truthtelling equilibrium"

cf. second-price auction

Def: truthful mechanism is a mechanism where "bid = value" is equilibrium.

Thm: if exists good mechanism, there exists good truthful mechanism

Proof: by picture

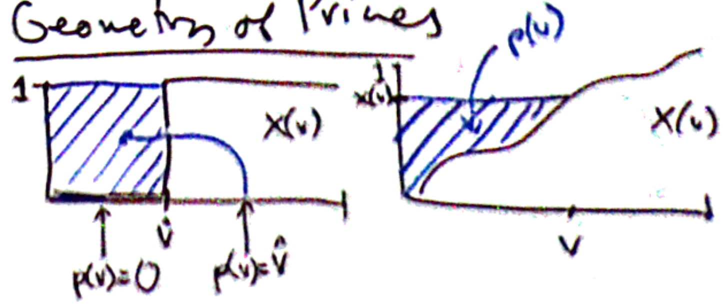


Conclusion: in theory, sufficient to look for optimal truthful mechanisms

Thm: truthful mechanism can implement allocation rule $x(\cdot)$ if and only if $x(\cdot)$ is monotone (non-decreasing).

- intuition 1: if x not monotone, high valued bidder could pretend to be low valued.
- intuition 2: can view monotone x like cumulative distribution func. of randomized posted price. (cdfs must be monotone)

Geometry of Prices



Welfare Maximization

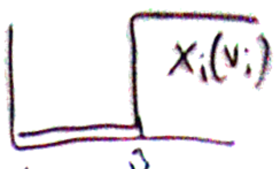
- e.g. single item auction
- second-price auction maximizes welfare in dominant strategy equilibrium

Framework for Optimization

0. relax truthfulness
1. optimize objective
2. check truthfulness (i.e. monotonicity)

Example: $k=2$ items, n bidders.

1. optimize objective:
   $\Rightarrow$ give two items to two highest valued bidders
2. check truthfulness: yes!
   - $\hat{v}_i$ = second highest of other values.
   - lose if $v_i < \hat{v}_i$
   - win if $v_i > \hat{v}_i$.
   - monotone



Q: what auction is this? (two items) third-price auction.

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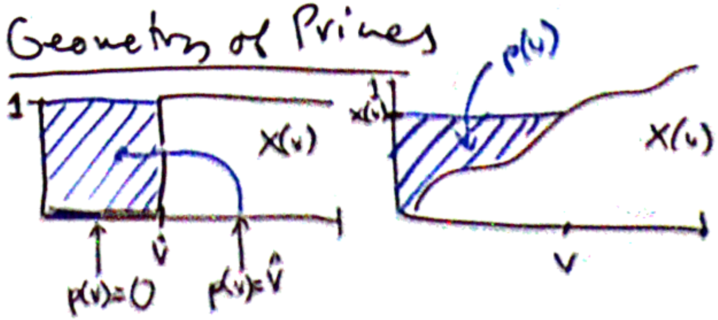

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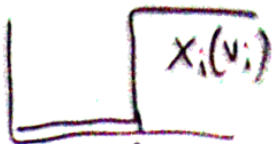

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Revenue Maximization

Recall: expected revenue* = expected virtual surplus

1: optimize objective:
   optimize virtual welfare.

2: check truthfulness
   $\Rightarrow X$ is monotone if ϕ is monotone.

Assume: ϕ's are monotone.

e.g. single item auction: allocate to bidder with highest non negative virtual value.

Example

- two bidders, $U[0,1]$ values: $\phi^{-1}(0) = 1/2$.
- $\phi(v) = 2v-1$
- bidder 1 wins: $\phi_1(v_1) > \max\{\phi_2(v_2), 0\}$
- $\phi_1(v) = \phi_2(v) = 2v-1$; $\Rightarrow \boxed{v_1 \geq \max(v_2, \phi^{-1}(0))}$

Q: what auction is this? A: SPA with reserve $\hat{v} = \phi^{-1}(0)$