CS 396: Online Markets

Lecture 13: Revenue Maximization (Cont.)

Last Time:

- revenue of auctions

Today:

- revenue of auctions (cont).
- virtual values.
- truthfulness and the revelation principle.
- optimization of truthful auctions.
- optimal first-price auctions.

Exercise: Allocation Rules

Recall:

- allocation rule: $x(\mathsf{v}) = \mathbf{Pr}[\text{bidder wins with value } \mathsf{v}]$
- probability of winning: $\mathbf{E}_{\mathsf{v}\sim F}[x(\mathsf{v})]$
- expected welfare: $\mathbf{E}_{\mathsf{v}\sim F}[\mathsf{v}\, x(\mathsf{v})]$

Setup:

- bidder's value is $\mathsf{v} \sim U[0,1]$
- allocation rules for mechanisms A and B

$$x_A(\mathsf{v}) = \mathsf{v} \qquad x_B(\mathsf{v}) = \begin{cases} 1 & \text{if } \mathsf{v} > 1/2 \\ 0 & \text{otherwise.} \end{cases}$$

Questions:

- what is probability that the bidder wins in A?
- what is expected welfare of A?
- what is probability that the bidder wins in B?
- what is expected welfare of B?

(Recall) Auction Revenue

Def: allocation rule:
$x(\mathsf{v}) = \mathbf{Pr}[\text{bidder wins with value } \mathsf{v}]$

Thm: revenue from x is $\mathbf{E}\left[\left[\mathsf{v} - \frac{1-F(\mathsf{v})}{f(\mathsf{v})}\right] x(\mathsf{v})\right]$.

Recall: **welfare** is value of winner,
expected welfare is $\mathbf{E}[\mathsf{v}\, x(\mathsf{v})]$

Def:

- **virtual value** is $\varphi(\mathsf{v}) = \mathsf{v} - \frac{1-F(\mathsf{v})}{f(\mathsf{v})}$
- **virtual welfare** is $\mathbf{E}[\varphi(\mathsf{v})\, x(\mathsf{v})]$

Conclusion: expected revenue = expected virtual welfare

Q: how to maximize virtual surplus?

A: allocate if $\varphi(\mathsf{v}) \geq 0$, i.e., if $\mathsf{v} \geq \varphi^{-1}(0)$

Example:

- $\mathsf{v} \sim U[0,1]$
- $F(\mathsf{v}) = \mathsf{v}$; $f(\mathsf{v}) = 1$
- $\varphi(\mathsf{v}) = \mathsf{v} - \frac{1-\mathsf{v}}{1} = 2\mathsf{v} - 1$

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- $\varphi^{-1}(0) = 1/2$

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- what is virtual surplus? 1/4.
- what mechanism does this? post price of 1/2

Corollary: optimal single-buyer mechanism posts price $\varphi^{-1}(0)$.

Q: multiple bidders?

Exercise: Optimal Pricing, Redux

Setup:

- you have one item to sell
- your value for keeping the item is 1.
- buyer with value from exponential distribution

Questions:

- what price should you offer to maximize your expected utility?

Revelation Principle

"may as well consider mechanisms with truthtelling equilibrium"

cf. second-price auction

Def: truthful mechanism is one where truthtelling is an equilibrium.

Thm: if exists good mechanism, exists good truthful mechanism.

Proof: by picture

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Conclusion: in theory, sufficient to look for optimal truthful mechanism.

Thm: truthful mechanism can implement allocation rule x iff x is monotonicly non-decreasing.

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- intuition 1: if it wasn't monotone, high-valued bidder would pretend to be low valued.
- intuition 2: cannot view non-monotone x as cdf of random price.

Geometry of Expected Payments:

- $p(\mathsf{v}) = \mathsf{v}x(\mathsf{v}) - \int_0^{\mathsf{v}} x(\mathsf{z})d\mathsf{z}$

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Welfare Maximization

- e.g., single-item auction
- second-price auction maximizes welfare in dominant strategy equilibrium

Framework for Optimization:

0. relax truthfulness
1. optimize objective
2. check truthfulness (a.k.a., monotonicity)

Example: $k = 2$ items, n bidders.

1. optimize objective
   give items to two bidders with highest values.
2. check truthfulness: yes
   - $\hat{\mathsf{v}}_i$ is second highest other bid.
   - i loses with bid below $\hat{\mathsf{v}}_i$
   - i wins with bid above $\hat{\mathsf{v}}_i$
   - $\Rightarrow$ monotone.

Q: what truthful auction has this outcome?

A: (two item) third-price auction.

Revenue Maximization

Thm: in multi-bidder mechanisms, expected revenue equals expected virtual welfare.

0. relax truthfulness
1. optimize virtual welfare
2. check truthfulness

Example: single-item auction

1. optimize virtual welfare:
   $\Rightarrow$ allocate to bidder with highest positive virtual value.
2. check truthfulness:
   $\Rightarrow$ monotone if φ are monotonicly non-decreasing.

e.g., bidder 1 wins if $\varphi_1(\mathsf{v}_1) \geq \max(\varphi_2(\mathsf{v}_2), 0)$,
$\Rightarrow$ 1 wins if $\mathsf{v}_1 \geq \max(\varphi_1^{-1}(\varphi_2(\mathsf{v}_2)), \varphi_1^{-1}(0))$

e.g., i.i.d. values (i.e., $\varphi_1 = \varphi_2 = \varphi$)
$\Rightarrow$ 1 wins if $\mathsf{v}_1 \geq \max(\mathsf{v}_2), \varphi^{-1}(0))$

Q: what auction does this?

A: second-price auction with reserve $\hat{\mathsf{v}} = \varphi^{-1}(0)$

Cor: for i.i.d. buyers second-price auction with reserve $\varphi^{-1}(0)$ is revenue optimal

Example: two bidders, values $U[0, 1]$

- $\varphi^{-1}(0) = 1/2$
- (from example) 5/12 is optimal revenue.

Revenue Optimal First-price Auction.

"truthful auctions are often impractical"

Approach: find first-price auction with same allocation rule as optimal truthful auction.

Example:

- two bidders, uniform values

Q: what is revenue optimal first-price auction?

A: first-price auction with reserve 1/2

- winner is bidder with highest value over 1/2
  $\Rightarrow$ x as second-price auction with reserve 1/2
  $\Rightarrow$ same expected virsual welfare