CS 396: Online Markets

Lecture 12: Revenue Maximization

Last Time:

- learning to bid (cont)
- partial feedback
- equilibrium of no-regret learning (coarse correlated equilibrium)

Today:

- ~~equilibrium of no-regret learning (coarse correlated equilibrium)~~
- revenue of auctions

Exercise: Optimal Pricing

Setup:

- you have one item to sell.
- buyer with value from exponential distribution
- exponential distribution cdf $F(z) = 1 - e^{-z}$

Questions:

- what price should you offer to maximize your expected revenue?

Auction Revenue

Recall:

- two bidders, values $U[0,1]$
- second-price auction
  - bids = values
  - $\mathbf{E}[\text{revenue}] = \mathbf{E}\big[\mathsf{b}_{(2)}\big] = \mathbf{E}\big[\mathsf{v}_{(2)}\big] = 1/3$
- first-price auction
  - bids = half of values
  - $\mathbf{E}[\text{revenue}] = \mathbf{E}\big[\mathsf{b}_{(1)}\big] = \mathbf{E}\big[\mathsf{v}_{(1)}/2\big] = 1/3$
- same revenue

Def: second-price auction with reserve $\hat{\mathsf{v}}$

- highest bidder with bid at least $\hat{\mathsf{v}}$ wins
- pays maximum of second-highest bid and $\hat{\mathsf{v}}$

Same thing as if seller entered a bid of $\hat{\mathsf{v}}$.

Example:

- two bidders, uniform values,
- reserve $\hat{\mathsf{v}} = 1/2$.
- revenue calculation:

Case	Probability	expected revenue
$\mathsf{v}_{(2)} < \mathsf{v}_{(1)} < \hat{\mathsf{v}}$	1/4	0
$\mathsf{v}_{(2)} < \hat{\mathsf{v}} < \mathsf{v}_{(1)}$	1/2	1/2
$\hat{\mathsf{v}} < \mathsf{v}_{(2)} < \mathsf{v}_{(1)}$	1/4	2/3

- total: $1/4 \times 0 + 1/2 \times 1/2 + 1/4 \times 2/3 = 5/12 > 1/3$

Note: by sometimes withholding the item, the seller can make more revenue.

Q: is there an auction with higher expected revenue?

A: no.

- Roger Myerson (1981) while at Kellogg
- 2007 Nobel prize in economics
- all you need to know is integration by parts!

Revenue Optimal Pricing

"optimal price to post a single buyer"

- buyer value $\mathsf{v} \sim F$
- revenue from price $\hat{\mathsf{v}}$: $\hat{\mathsf{v}}\mathbf{Pr}[\mathsf{v} \geq \hat{\mathsf{v}}] = \hat{\mathsf{v}}(1 - F(\hat{\mathsf{v}}))$
- to optimize $\frac{d}{d\hat{\mathsf{v}}}[\hat{\mathsf{v}}(1 - F(\hat{\mathsf{v}}))] = 1 - F(\hat{\mathsf{v}}) - \hat{\mathsf{v}} f(\hat{\mathsf{v}}) = 0$
- note: tradeoff $\hat{\mathsf{v}}$ with probability of sale $1 - F(\hat{\mathsf{v}})$.

Example:

- $\mathsf{v} \sim U[0,1]$, i.e., $F(\mathsf{z}) = \mathsf{z}$
- $\frac{d}{d\hat{\mathsf{v}}}[\hat{\mathsf{v}}(1 - \hat{\mathsf{v}})] = 1 - 2\hat{\mathsf{v}} = 0$
- $\Rightarrow \hat{\mathsf{v}} = 1/2$.
- expected revenue is $1/2 \times (1 - F(1/2)) = 1/4$.

Exercise: Pricing Lotteries

Setup:

- buyer with value $U[0,1]$
- menu of options:
  (a) price of 0: receive nothing
  (b) price of 1/6: receive item with probability 1/2
  (c) price of 1/2: receive item with probability 1

Questions:

- what value of buyer is indifferent between (a) and (b)?
- what value of buyer is indifferent between (b) and (c)?
- what is expected revenue when buyer buys preferred option?

Revenue Analysis of Lotteries

"e.g., in multi-bidder auction, outcome depends on other bids"

Def: allocation rule:
$x(\mathsf{v}) = \mathbf{Pr}[\text{bidder with value } \mathsf{v} \text{ wins}]$

Def: posted pricing implementation of x:

- draw price $\hat{\mathsf{v}}$ from distirbution $G(\mathsf{z}) = x(\mathsf{z})$
- $x(\mathsf{z}) = \mathbf{Pr}[\text{bidder offered price } \hat{\mathsf{v}} \leq \mathsf{v}] = G(\mathsf{z})$

Lemma: revenue from x is $\int_0^\infty \mathsf{z}(1 - F(\mathsf{z}))x'(\mathsf{z})d\mathsf{z}$.

Proof:

- density function for $\hat{\mathsf{v}}$ is $x'(\hat{\mathsf{v}})$.
- revenue if offer price $\hat{\mathsf{v}}$ is $\hat{\mathsf{v}}(1 - F(\hat{\mathsf{v}}))$
- definition of expectation.

Thm: revenue from x is $\mathbf{E}\left[\left[\mathsf{v} - \frac{1-F(\mathsf{v})}{f(\mathsf{v})}\right] x(\mathsf{v})\right]$.

Recall: integration by parts

$\int_a^b f(z)\, g'(z)\, dz = [f(z)g(z)]_a^b - \int_a^b f'(z)\, g(z)\, dz$

Proof:

- integration by parts:
  $\int_0^\infty \mathsf{z}(1 - F(\mathsf{z}))x'(\mathsf{z})d\mathsf{z}$
  $= [\mathsf{z}(1 - F(\mathsf{z}))x(\mathsf{z})]_0^\infty - \int_0^\infty [1 - F(\mathsf{z}) - \mathsf{z}\, f(\mathsf{z})]\, x(\mathsf{z})\, d\mathsf{z}$
- simplify with $\mathsf{z}(1 - F(\mathsf{z})) = 0$ at $\mathsf{z} = 0$ and $\mathsf{z} = \infty$
- factor out density to write as expectation:
  $\mathbf{E}_{\mathsf{v}\sim F}\left[\left[\mathsf{v} - \frac{1-F(\mathsf{v})}{f(\mathsf{v})}\right] x(\mathsf{v})\right]$

Recall: welfare is value of winner,
expected welfare is $\mathbf{E}[\mathsf{v}\, x(\mathsf{v})]$

Def:

- **virtual value** is $\varphi(\mathsf{v}) = \mathsf{v} - \frac{1-F(\mathsf{v})}{f(\mathsf{v})}$
- **virtual welfare** is $\mathbf{E}[\varphi(\mathsf{v})\, x(\mathsf{v})]$

Conclusion: expected revenue = expected virtual welfare

Q: how to maximize virtual surplus?

A: allocate if $\varphi(\mathsf{v}) \geq 0$, i.e., if $\mathsf{v} \geq \varphi^{-1}(0)$

Corollary: optimal single-buyer mechanism posts price $\varphi^{-1}(0)$.

Conclusion: selling lotteries does not improve revenue.

Example:

- $\mathsf{v} \sim U[0,1]$
- $F(\mathsf{v}) = \mathsf{v}$; $f(\mathsf{v}) = 1$
- $\varphi(\mathsf{v}) = \mathsf{v} - \frac{1-\mathsf{v}}{1} = 2\mathsf{v} - 1$

DRAW PICTURE

- $\varphi^{-1}(0) = 1/2$

A: optimal mechanism

Q: what is virtual surplus? **A:** 1/4.

Q: what mechanism does this? **A:** post price of 1/2

Multi-bidder Mechanisms

Thm: in multi-bidder mechanisms, expected revenue equals expected virtual welfare.

Q: how to maximize virtual welfare in single-item auction?

A: allocate to bidder with highest positive virtual value.

e.g., bidder 1 wins if $\varphi_1(\mathsf{v}_1) \geq \max(\varphi_2(\mathsf{v}_2), 0)$,
$\Rightarrow$ 1 wins if $\mathsf{v}_1 \geq \max(\varphi_1^{-1}(\varphi_2(\mathsf{v}_2)), \varphi_1^{-1}(0))$

e.g., i.i.d. values (i.e., $\varphi_1 = \varphi_2 = \varphi$)
$\Rightarrow$ 1 wins if $\mathsf{v}_1 \geq \max(\mathsf{v}_2), \varphi^{-1}(0))$

Q: what auction does this?

A: second-price auction with reserve $\hat{\mathsf{v}} = \varphi^{-1}(0)$

Cor: for i.i.d. buyers second-price auction with reserve $\varphi^{-1}(0)$ is revenue optimal

Example: two bidders, values $U[0, 1]$

- $\varphi^{-1}(0) = 1/2$
- (from example) $5/12$ is optimal revenue.

Equilibrium of No-regret Learning

"outcomes of games under learning"

Def: coarse correlated equilibrium (CCE)

- mediator offers joint distributions of actions (r, c)
- players either
  - follow mediator
  - pick any fixed outcome
- CCE if best response is to follow mediator

Thm: play is no-regret iff emprical distribution of play is CCE.

- fix sequence $((r^0, c^0), \ldots, (r^n, c^n))$
- **no-regret** for player R, for all actions r^* of R: $\sum_i R_{r^i,c^i} \geq \sum_i R_{r^*,c^i}$
- consider mediator:
  - pick i uniformly from round $\{1, \ldots, n\}$
  - recommend r^i to R and c^i to C.
- no-regret for R and C $\Leftrightarrow$ mediator is CCE.