Auction Revenue

Recall:

- two bidders, $U[0,1]$ values
- second-price auction
  - bids = values
  - $E[\text{revenue}] = E[b_{(2)}] = E[v_{(2)}] = \frac{1}{3}$
- first-price auction
  - bids = half of values
  - $E[\text{revenue}] = E[b_{(1)}] = E[v_{(1)}/2] = \frac{1}{2}E[v_{(1)}] = \frac{1}{2}\cdot\frac{2}{3} = \frac{1}{3}$
- same revenue.

Q: how to get more revenue?

Def: second-price auction with reserve $\hat{v}$.

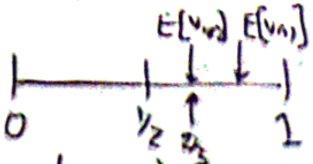
- highest bidder with bid at least $\hat{v}$
- winner (if any) pays max of highest other bid and $\hat{v}$.

Example:

- 2 bidders, $U[0,1]$ values
- reserve $\hat{v} = \frac{1}{2}$
- revenue calculation:

	case	prob	expected revenue
Case 1	$v_{(2)} < v_{(1)} < \hat{v}$	$1/4$	0
Case 2	$v_{(2)} < \hat{v} < v_{(1)}$	$1/2$	$1/2$
Case 3	$\hat{v} < v_{(2)} < v_{(1)}$	$1/4$	$E[v_{(2)} \mid v_{(2)}, v_{(1)} \geq 1/2] = 2/3$

Case 3 analysis:



- total: $\frac{1}{4}\cdot 0 + \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{4}\cdot\frac{2}{3} = \frac{5}{12} > \frac{1}{3}$.

Note: seller can make more revenue with reserve (and sometimes not selling)

Q: is there an auction with higher revenue?

A: no (reserves are good for optimizing revenue)

- Roger Myerson (1981)
- Kellogg faculty
- 2007 Nobel in economics
- integration by parts.

Revenue Optimal Pricing

"optimal price to post to a single buyer"

Model

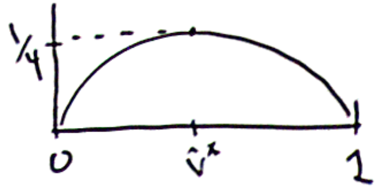
- buyer value $v \sim F$.
- revenue from price $\hat{v}$:

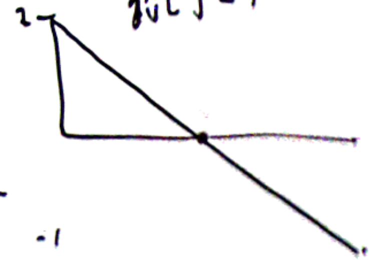
  $\hat{v} \cdot \Pr[v > \hat{v}] = \hat{v} \cdot (1 - F(\hat{v}))$
- optimize: $\underbrace{\frac{d}{d\hat{v}}\left[\hat{v}\,(1 - F(\hat{v}))\right]}_{1 - F(\hat{v}) - \hat{v} f(\hat{v})} = 0$ $f(z)$ is density function

Example

- $v \sim U[0,1]$, $F(z) = z$; $f(z) = 1$
- expected revenue: $\hat{v}(1 - \hat{v}) = \hat{v} - \hat{v}^2$

  $\frac{d}{d\hat{v}}[\cdot] = 1 - 2\hat{v}$





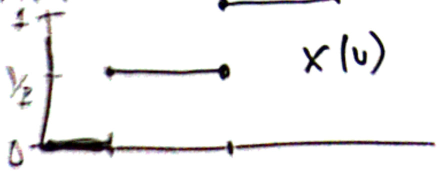
- $\hat{v}^* = 1/2$.
- expected revenue: $1/2\,(1 - 1/2) = 1/2 \cdot 1/2 = 1/4$

Revenue Analysis of Lotteries

"e.g. in multi-bidder auctions, chance of winning depends on other bids (which are random)"

Def: allocation rule

$x(v) = \Pr[\text{buyer with value } v \text{ wins}]$



Def: posted pricing implementation of $x(v)$

- draw price $\hat{v} \sim G(z)$
- $G(z) = \Pr[\text{bidder is offered price } \hat{v} < z]$

$= \Pr[\text{bidder wins with value } z] = x(z)$

Lemma: revenue from x when $v \sim F$ is

$$E[\text{revenue}] = \int_0^\infty z(1-F(z))\,x'(z)\,dz$$

Proof:

- view x as random price $\hat{v} \sim G$
- $E[\text{revenue}] = E_{\hat{v}\sim G}[\hat{v}(1-F(\hat{v}))]$

$$= \int_0^\infty \hat{v}(1-F(\hat{v}))\,g(\hat{v})\,d\hat{v}$$

$g(v) = x'(v)$

- $E[\text{revenue}] = \int_0^\infty \hat{v}(1-F(\hat{v}))\,x'(\hat{v})\,d\hat{v}$

Thm: revenue from $v \sim F$ of x is

$$E[\text{revenue}] = E_{v\sim F}\left[\left(v - \frac{1-F(v)}{f(v)}\right)x(v)\right]$$

Recall Integration By Parts

$$\int_a^b f(z)\,g'(z)\,dz = [f(z)\,g(z)]_a^b - \int_a^b f'(z)\,g(z)\,dz$$

Proof:

- integration by parts:

$$E[\text{revenue}] = [z(1-F(z))\,x(z)]_0^\infty - \int_0^\infty [1-F(z) - zf(z)]\,x(z)\,dz$$

- simplify $z(1-F(z)) = 0$ at $z=0$ & $z=\infty$
- write like expectation of $v \sim F$

$$E[\text{revenue}] = \int_0^\infty \left(z - \frac{1-F(z)}{f(z)}\right)x(z)\,f(z)\,dz$$

Recall: welfare is value of winner

expected welfare is $E[v\,x(v)]$

Def: virtual value $\varphi(v) = v - \frac{1-F(v)}{f(v)}$

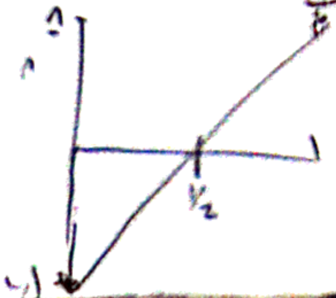
virtual welfare $E[\varphi(v)\,x(v)]$

Conclusion: E[revenue] = E[virtual welfare]

Q: how to optimize virtual welfare?

Example

- $v \sim U[0,1]$, $F(v) = v$; $f(v) = 1$
- $\varphi(v) = v - \frac{1-v}{1} = 2v - 1$



give buyer item
if v.v. is positive
if $\varphi(v) > 0$.
i.e. $v > \varphi^{-1}(0) = 1/2$
post price of 1/2.

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$$g(v) = x'(v)$$

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Proof:

- integration by parts:

$$E[\text{revenue}] = [z(1-F(z))x(z)]_0^\infty - \int_0^\infty [1-F(z) - z f(z)]\,x(z)\,dz$$

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$$E[\text{revenue}] = \int_0^\infty \left(z - \frac{1-F(z)}{f(z)}\right)x(z)\,f(z)\,dz$$