Auction Theory

"predict outcomes of auctions, which auctions are better?"

Example Goal:

- First-price auction (FPA)
  - highest-bidder wins (random tie breaking)
  - winner pays bid
- Second-price auction (SPA)
  - highest-bidder wins
  - winner pays second highest bid.
- higher welfare? (value of winner)
- higher revenue? (payments to auctioneer)

Consider ~~(31, 20)~~ Not Nash

$\boxed{(61, 61)}$ Yes! 11 ~~(21, 20)~~ Not Nash

Recall: SPA

- "bid = value" is dominant-strategy (DSE)
- e.g. two bidders
  - $v_1 = 90$; $v_2 = 30$
  - in DSE, $b_1 = 90$; $b_2 = 30$
- bidder 1 wins, pay 30
- welfare: 90
- revenue: 30

Nash Equilibria of FPA

"analyze as complete information game"

Example: FPA

- two bidders, action space $\{0, \ldots, 100\}$
- values are known
- e.g. $v_1 = 90$; $v_2 = 30$

Q: What are Nash equilibria?

A: $(31, 30)$ and $(30, 29)$ $= \{(31, x) : x \geq 29\}$

Q: is $(30, 30)$ a Nash? A: No.

Thm: in Nash eq. of discrete FPA

- highest valued bidder wins.
- winner pays second-highest value + (0 or min bid increment) if no overbidding.

Conclusion: with full information

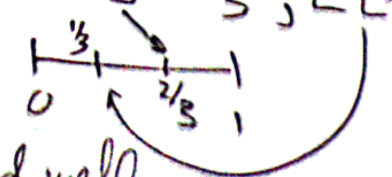
FPA & SPA have approx same outcome.

Incomplete Information

"bidders values are random from a known distribution"

Example: SPA

- two bidders, values $U[0,1]$
- analysis:
  - "bid = value" is DSE
  - $E[v_{(1)}] = 2/3$; $E[v_{(2)}] = 1/3$



- expected welfare: $E(v_{(1)}] = 2/3$
- expected revenue: $E(v_{(2)}] = 1/3$.

Bayes-Nash Equilibrium

"how do bidders bid when no DSE?"

Example: FPA

- two bidders, values $U[0,1]$

Q: what is Equilibrium?

A: "guess and verify"

- guess bidder 2 bids half of value. $(b_2 = v_2/2)$
- how should bidder 1 bid?
- plan: for bidder 1
  - write winning prob as func of bid.
  - write utility as func of v_1 & b_1
  - solve for optimal bid.
- winning probability

$$\Pr[1 \text{ wins with bid } b_1] = \Pr[b_2 < b_1]$$
$$= \Pr[v_2/2 < b_1] = \Pr[v_2 \le 2b_1]$$
$$= F_{v_2}(2b_1) = 2b_1$$

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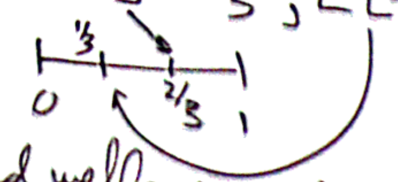

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- utility

$$u(v_1, b_1) = (v_1 - b_1) \cdot \Pr[1 \text{ wins w. bid } b_1]$$
$$= (v_1 - b_1) \cdot 2b_1$$
$$= 2v_1 b_1 - 2b_1^2$$

- optimal bid

$$\frac{d}{db_1}\left[2v_1 b_1 - 2b_1^2\right] = 2v_1 - 4b_1 = 0$$

$$\Rightarrow b_1 = v_1/2.$$

- conclusion
  - assumed $b_2 = v_2/2$
  - showed $b_1 = v_1/2$
  - bidding half of value is Equilibrium
- expected welfare: $E[v_{(1)}] = 2/3$.
- expected revenue: $E[v_{(1)}/2] = 1/3$.

Thm: iid FPA & SPA have same welfare & revenue in Bayes-Nash Eq.

Def: bidders bidding with **common prior** know distribution of values: $\mathbf{v} \sim \mathbf{F}$.

Notation: $\mathbf{v}_{-i} = (v_1, \ldots v_{i-1}, ?, v_{i+1}, \ldots, v_n)$

Def: a strategy profile $\sigma = (\sigma_1, \ldots, \sigma_n)$ (σ_i maps v_i to bid b_i) is Bayes-Nash Equilibrium (BNE) if for all i, for all v_i, $b_i = \sigma_i(v_i)$ is best response to other bidder bid $\sigma_{-i}(\mathbf{v}_{-i})$, and $\mathbf{v}_{-i} \sim \mathbf{F}_{-i}$.

Example $\sigma = (\sigma, \sigma)$ with $\sigma(v) = v/2$ is BNE for 2 iid $U[0,1]$ bidders in FPA.

- utility

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