CS 396: Online Markets

Lecture 8: Equilibria in Auctions

Last Time:

- game theory
- bimatrix games
- Nash equilbrium
- dominant strategy equilibrium

Today:

- auction theory
- second-price auction
- first-price auction
- complete information analysis (Nash equilibrium)
- incomplete information analysis (Bayes-Nash equilbrium)

Exercise: Pretty Puzzle

Setup:

- you are playing a game against your n classmates.
- pick an integer between 0 and 100
- the students who pick the number closest to 1/2 the average wins.

Questions:

- play the game!
- Identify an action that is in a Nash equilibrium.

Auction Theory

"predict outcomes in auctions; which auctions are better"

Example:

- first-price auction (FPA)
  - highest bidder wins (random tie-breaking)
  - winner pays bid
- second-price auction (SPA)
  - highest bidder wins (random tie-breaking)
  - winner pays-second highest bid
- which auction has higher welfare (value of winner)
- which auction has higher revenue? (payment to auctioneer)

Recall:

- second price auction
- "bid = value" is dominant strategy
- e.g, two bidders,
  - $\mathsf{v}_1 = 90$, $\mathsf{v}_2 = 30$
  - in DSE, $\mathsf{b}_1 = 90$, $\mathsf{b}_2 = 30$
- bidder 1 wins, pays 30.
- welfare is 90, revenue is 30.

Nash Equilibria of First-price Auction

"analyze as a **complete information** game"

Example:

- FPA, two bidder, action space: $\{0, \ldots, 100\}$
- values known
- e.g., $\mathsf{v}_1 = 90$, $\mathsf{v}_2 = 30$

Q: what are the Nash equilibria?

A: $(31, 30)$ and $(30, 29)$

Q: is $(30, 30)$ a Nash? **A:** No.

Thm: in Nash eq. of discrete FPA

- highest-valued agent wins
- winner pays second-highest value or second-highest + minimum bid increment.

Conclusion: with full information FPA and SPA have approximately the same outcome.

Exercise: Winning Probabilities

Recall:

- cumulative distribution function: $F_X(\mathsf{z}) = \mathbf{Pr}[X < \mathsf{z}]$
- uniform distribution on $[0,1]$: $F_X(\mathsf{z}) = \mathsf{z}$
- first-price auction: highest bidder wins, winner pays bid.
- independent and identical distributions (i.i.d.):
  - $X_1, \ldots, X_n \sim F_X$
  - $\mathbf{X}_{-i} = (X_1, \ldots, X_{i-1}, ?, X_{i+1}, \ldots, X_n)$
  - $\mathbf{Pr}[X_i < \mathsf{z} \mid \mathbf{X}_{-i}] = \mathbf{Pr}[X_i < \mathsf{z}]$

Setup:

- you are bidding in a first-price auction
- other bidders with i.i.d. uniform bids on $[0,1]$

Questions: If you bid $\mathsf{b} = 1/2$,

- What is the probability you win against one other bidder?
- What is the probability you win against two other bidders?

Incomplete Information

"bidders values are random from a known distribution"

Example: second-price auction

- two bidders, values $U[0,1]$
- analysis:
  - "bid = value" is dominant strategy
  - $\mathbf{E}[\mathsf{v}_{(1)}] = 2/3$, $\mathbf{E}[\mathsf{v}_{(2)}] = 1/3$
- expected welfare: 2/3.
- expected revenue: 1/3.

Bayes-Nash Equilibrium

"how do bidders bid when no DSE"

Example:

- first-price auction
- two bidders, values $U[0,1]$

Q: what is equilibrium?

A: (guess and verify)

- suppose bidder 2 bids half of value
- how should you bidder 1 bid?
- plan:
  - write winning probability as function of b
  - write utility as function of v and b
  - solve for optimal bid.
- winning probability:

$$\begin{aligned}\mathbf{Pr}[\text{win with bid } \mathsf{b}] &= \mathbf{Pr}[\mathsf{b}_2 < \mathsf{b}] \\ &= \mathbf{Pr}[\mathsf{v}_2/2 < \mathsf{b}] \\ &= \mathbf{Pr}[\mathsf{v}_2 < 2\,\mathsf{b}] \\ &= F(2\,\mathsf{b}) \\ &= 2\,\mathsf{b}\end{aligned}$$

- utility:

$$\begin{aligned}\mathsf{u}(\mathsf{v},\mathsf{b}) &= (\mathsf{v}-\mathsf{b})\,\mathbf{Pr}[\text{win with bid } \mathsf{b}] \\ &= (\mathsf{v}-\mathsf{b})\,2\,\mathsf{b} \\ &= 2\,\mathsf{v}\,\mathsf{b} - 2\,\mathsf{b}^2\end{aligned}$$

- optimal bid:
  - $\frac{d}{d\mathsf{b}}[\mathsf{u}(\mathsf{v},\mathsf{b})] = 2\,\mathsf{v} - 4\,\mathsf{b} = 0$
  - $\mathsf{b} = \mathsf{v}/2$
- conclusion:
  - assumed bidder 2 bids half of value
  - showed that bidder 1 bids half of value
  - "bid half of value" is equilibrium.
- expected welfare: $\mathbf{E}[\mathsf{v}_{(1)}] = 2/3$
- expected revenue: $\mathbf{E}[\mathsf{b}_{(1)}] = \mathbf{E}[\mathsf{v}_{(1)}/2] = 1/3$

Def: bidders with **common prior** know distribution of values $\mathbf{v} \sim \boldsymbol{F}$

Notation: $\mathbf{v}_{-i} = (\mathsf{v}_1, \ldots, \mathsf{v}_{i-1}, ?, \mathsf{v}_{i+1}, \ldots, \mathsf{v}_n)$

Def: strategy profile $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n)$ (σ_i maps value v_i to bid b_i) is **Bayes-Nash equilibrium (BNE)** if for all i, $\sigma_i(\mathsf{v}_i)$ is best response when other agents play $\boldsymbol{\sigma}_{-i}(\mathbf{v}_{-i})$ with $\mathbf{v}_{-i} \sim \boldsymbol{F}_{-i}$.

Claim: $\boldsymbol{\sigma} = (\sigma, \sigma)$ with $\sigma(\mathsf{v}) = \mathsf{v}/2$ is a BNE of 2-bidder FPA with values i.i.d. $U[0,1]$
