Game Theory

"predict outcomes of strategic scenarios"

Def: bimatrix game.

- two player game
- row player with n actions
- column player has m actions
- payoff matrices $R, C \in \mathbb{R}^{n \times m}$
- on row action i, column action j
  - row player payoff R_{ij}
  - column player payoff C_{ij}

Example: Chicken $(n=m=2)$

	Stay	Swerve
Stay	-5,-5	1,-1
Swerve	-1,1	-1,-1

Equilibria

"stable outcomes in a game"

Def best response

- if column player plays j
- row player's best response is $i^* = \arg\max_i R_{ij}$

(vice versa for column player)

Q: in Chicken

- what should R do if C stays?
  - A: R should swerve
- what should C do if R swerves?
  - A: C should stay

Conclusion (Swerve, Stay) is a mutual best response.

A.k.a. pure Nash equilibrium

Symmetry: (Stay, Swerve) is pure Nash

Randomized Equilibria

"a.k.a. <u>mixed</u> strategies"

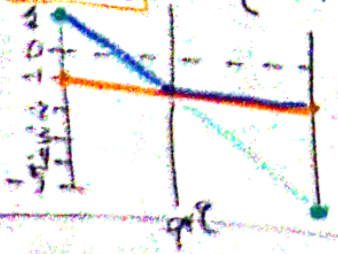
Q: are there other equilibria?

A: yes

- suppose C stays w. prob. q and swerve w. prob. $1-q$.
- R's payoffs

$\boxed{\text{stay}}$: $-5q + 1(1-q) = 1-6q$

$\boxed{\text{Swerve}}$: $-q + -1(1-q) = -1$



$1-6q^* = -1$

$2 = 6q^*$

$\frac{1}{3} = q^*$

- R is indifferent at $q^* = \frac{1}{3}$

<u>Example</u>: Chicken $(n=m=2)$

	Stay	Swerve
Stay	$-5,-5$	$1,-1$
Swerve	$-1,1$	$-1,-1$

<u>Conclusion</u> $p^* = \frac{1}{3}$, $q^* = \frac{1}{3}$

row player prob of stay — column player prob of stay

is a <u>mixed Nash equilibrium</u>

<u>Def</u>: n-dimensional probability simplex

$$\Delta_n = \{ p \in [0,1]^n : \sum p_i = 1 \}$$

<u>Def</u>: a <u>mixed Nash equilibrium</u> is

- row player plays $p^* \in \Delta_n$
- column player plays $q^* \in \Delta_m$
- p^* & q^* are mutual best response.
  - p^* is best response to q^*: $p^* \in \operatorname{argmax}_p p^T R q^*$
  - q^* " " " p^*: $q^* \in \operatorname{argmax}_q p^{*T} C q$

<u>Fact</u>: (p^*, q^*) is Nash iff

- $p_i^* > 0 \Rightarrow$ "action i is best response to q^*"
- $q_j^* > 0 \Rightarrow$ "action j is best response to p^*"

<u>Note</u>: use this fact to find mixed Nash.

Existence of Equilibria

Example: Hide and Seek (a.k.a. "Matching Pennies")

	Seek at A	Seek at B
Hide at A	-1, 1	1, -1
Hide at B	1, -1	-1, 1

Q: pure Nash? A: no

Q: mixed Nash? A: yes

- R plays $p^* = (\frac{1}{2}, \frac{1}{2}) \Rightarrow$ C indiff.
- C plays $q^* = (\frac{1}{2}, \frac{1}{2}) \Rightarrow$ R indiff.

$\Rightarrow$ nash eq.

Thm. [Nash '51] (possibly mixed) Nash equilibria always exist (games with finite action spaces)

Dominant Strategy Equilibrium

Example: Prisoner's Dilemma

	Defect	Cooperate
Defect	-2, -2	0, -3
Cooperate	-3, 0	-1, -1

Q: in Prisoner's Dilemma

- What should R do if C Defects?
  A: R should Defect
- What should R do if C Cooperates?
  A: R should Defect.

Conclusion:

- Defect is dominant strategy
- (Defect, Defect) is dominant strategy equilibrium (DSE)

Note

- DSE $\subseteq$ pure Nash $\subseteq$ Nash.

Challenge: prediction with multiple equilibria.

Dominant Strategy Equilibrium

Example: Prisoner's Dilemma

	Defect	Cooperate
Defect	-2,-2	0,-3
Cooperate	-3,0	-1 -1

Q: in Prisoner's Dilemma

- What should R do if C Defects?
  A: R should Defect
- what should R do if C Cooperates?
  A: R should Defect.

Conclusion:

- Defect is dominant strategy
- (Defect, Defect) is dominant strategy equilibrium (DSE)