Recall: Online Allocation

"buyers show up one at a time, must make decision for each buyer before next buyer arrives."

A.k.a online gambling

Assumption: values drawn from known distribution.

$v_1 \sim F_1, \ldots, v_n \sim F_n$

Example: $n = 2$ buyers, values $U[0,8]$

- thresholds $\hat{v} = (4, 0)$
- $E[\text{payoff}] = 5$

$$E[\text{payoff}] = E[v_1 \mid v_1 > \hat{v}_1]\Pr[v_1 > \hat{v}_1] + E[v_2 \mid v_1 < \hat{v}_1]\Pr[v_1 < \hat{v}_1]$$

$$-E[v_1 \mid v_1 > \hat{v}_1] = 6 \quad (U[0,8],\ \hat{v}_1 = 4)$$

$$-\Pr[v_1 > \hat{v}_1] = 1/2$$

$$-E[v_2 \mid v_1 < \hat{v}_1] = E[v_2] = 4$$

$$-\Pr[v_1 < \hat{v}_1] = 1 - \Pr[v_1 > \hat{v}_1] = 1/2$$

$$\Rightarrow 6 \times 1/2 + 4 \times 1/2 = 10 \times 1/2 = 5$$

Approximation

"comparing online alg to optimal offline algorithms"

Defn: an online alg ALG is a β-approximation to the optimal offline alg OPT if

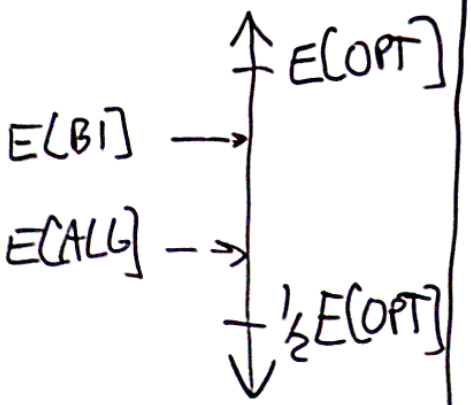
$$E[ALG(v)] \geq \frac{1}{\beta} E[OPT(v)]$$

Note: $\beta \geq 1$; smaller is better.

Aka "competitive analysis"

Recall Alg: backwards induction.

1. expected payoff from no prize is
$$P_{n+1} = 0$$
2. iterate from $i = n$ to 1:
   a) threshold for prize i is:
   $$\hat{v}_i = P_{i+1}$$
   b) expected payoff from $\{i, \ldots, n\}$
   $$P_i = E[\max(v_i, P_{i+1})]$$
3. output thresholds
$$\hat{v} = (\hat{v}_1, \ldots, \hat{v}_n)$$



The eBay Problem

"posting a uniform price"

Thm: The gambler can use a single threshold $\hat{v}$ in obtain expected payoff at least half of expected optimal offline payoff, i.e. a 2-approximation.

$$E[ALG(v_i)] \geq \tfrac{1}{2} E(\max_i v_i) = \tfrac{1}{2} E[OPT(v_i)]$$

Aka prophet inequality.

Fact: $E[OPT(v_i)] \geq E(BI(v_i)) \geq E[ALG(v_i)] \geq \tfrac{1}{2} E(OPT(v_i))$

Proof:

0. Notation: (we will choose $\hat{v}$ later.)
   - $\hat{v}$ threshold.
   - $(\cdot)^+ = \max(\cdot, 0)$
   - $x = \Pr[\text{no prize}] = \prod_i \underbrace{\Pr[v_i < \hat{v}]}_{F_i(\hat{v})}$
   - $\mathcal{E}_i$ = "event that $\forall j \neq i$ $v_j < \hat{v}$"

1. upper bound on prophet:

$$\begin{aligned}
E[OPT(v)] &= E[\max_i v_i] \\
&= \hat{v} + E[\max_i (v_i - \hat{v})] \\
&\le \hat{v} + E[\max_i (v_i - \hat{v})^+] \\
&\le \hat{v} + E[\textstyle\sum_i (v_i - \hat{v})^+] \\
&= \hat{v} + \textstyle\sum_i E[(v_i - \hat{v})^+]
\end{aligned}$$

2. lower bound on gambler

$$\begin{aligned}
E[ALG(v)] &= (1-x)\hat{v} \\
&\quad + E[\text{amount selected prize exceeds } \hat{v}] \\
&\ge (1-x)\hat{v} + E[\text{amount selected prize exceeds } \hat{v} \text{ when only one such prize}] \\
&\ge (1-x)\hat{v} + \sum_i E[(v_i - \hat{v})^+ \mid \mathcal{E}_i] \Pr[\mathcal{E}_i] \qquad \{v_j, j \neq i\}
\end{aligned}$$

(a) $\Pr[\mathcal{E}_i] = \prod_{j \neq i} \Pr[v_j < \hat{v}] \ge x$

(b) $E[(v_i - \hat{v})^+ \mid \mathcal{E}_i] = E[(v_i - \hat{v})^+]$

$$\ge \underbrace{(1-x)}_{1/2}\hat{v} + \underbrace{x}_{1/2} \textstyle\sum_i E[(v_i - \hat{v})^+]$$

3. combine with $x = 1/2$.

"choose $\hat{v}$ so that $\Pr[\text{no prize}] = x = 1/2$"

$$\Rightarrow E[ALG(v)] \ge \tfrac{1}{2} E[OPT(v)].$$