CS 396: Online Markets

Lecture 3: Online Approximation

Last Time:

- course philosophy
- second price auction (cont)
- online allocation: backwards induction
- online mechanisms: sequential pricing

Today:

- online allocation (cont)
- approximation
- prophet inequality
- secretary problem

Exercise: Approximation Ratio

Setup:

- n=2 buyers
- values $U[0,1]$

Question: What is the ratio of the expected welfares of the optimal "offline" algorithm to the online backwards induction algorithm?

Recall: Online Allocation

"buyers show up one at a time, must make decision for each buyer before next buyer arrives"

A.k.a. "Online Gambling"

Assumption 1: values are drawn from known probability distributions

Recall Alg: backward induction

1. expected payoff from no prize is P_{n+1}
2. Iterate from $i = n$ to 1
   a. threshold for prize i is $\hat{v}_i = P_{i+1}$
   b. expeted payoff from games $\{i, \ldots, n\}$ $P_i = \mathbf{E}[\max(v_i, P_{i+1})]$
3. output thresholds $\hat{\mathbf{v}} = (\hat{v}_1, \ldots, \hat{v}_n)$

Example: $n = 2$ buyers, values $U[0, 8]$

- thresholds: $\hat{\mathbf{v}} = (4, 0)$
- welfare: 5/8
  - $\mathbf{E}[v_1 \mid v_1 > \hat{v}_1]\mathbf{P}r[v_1 > \hat{v}_1]$
    $+ \mathbf{E}[v_2 \mid v_1 < \hat{v}_1]\mathbf{P}r[v_1 < \hat{v}_1]$
  - $\mathbf{P}r[v_1 > \hat{v}_1] = \mathbf{P}r[v_1 < \hat{v}_1] = 1/2$
  - $\mathbf{E}[v_1 \mid v_1 > \hat{v}_1] = 6$
  - $\mathbf{E}[v2 \mid v_1 < \hat{v}_1] = \mathbf{E}[v_2] = 4.$
  - $\Rightarrow 6(1/2) + 4(1/2) = 5$

Comparing online algorithm to optimal offline algorithm

Defn: an online algorithm ALG is a β-approximation to the optimal offline algorithm OPT if

$$\mathbf{E}[\mathrm{ALG}(\mathbf{v})] \geq \tfrac{1}{\beta}\mathbf{E}[\mathrm{OPT}(\mathbf{v})]$$

A.k.a. "competitive analysis"

Note: $\beta \geq 1$ and smaller is better.

The eBay Problem

"posting a uniform price"

Thm: gambler can used single threshold $\hat{v}$ and obtain expected payoff at least half of expected optimal offline payoff, i.e., 2-approximation.

A.k.a. prophet inequality

"optimal offline payoff" is prophet's payoff (who can see future)"

Proof:

0. Notation
   - $\hat{v}$ = threshold
   - $(\cdot)^+ = \max(\cdot, 0)$
   - $\chi = \prod_i \mathbf{P}r[v_i < \hat{v}] = \mathbf{P}r[\text{no prize}]$
   - event $\mathcal{E}_i$ = "$\forall j \neq i,\ v_j < \hat{v}$"

1. upperbound on prophet

   $\mathbf{E}[\mathrm{OPT}(\mathbf{v})]$

   $$\begin{aligned}
   &= \mathbf{E}[\max_i v_i] \\
   &= \hat{v} + \mathbf{E}[\max_i (v_i - \hat{v})] \\
   &\leq \hat{v} + \mathbf{E}[\max_i (v_i - \hat{v})^+] \\
   &\leq \hat{v} + \mathbf{E}[\sum_i (v_i - \hat{v})^+] \\
   &= \hat{v} + \sum_i \mathbf{E}[(v_i - \hat{v})^+]
   \end{aligned}$$

2. lowerbound on gambler

   $\mathbf{E}[\mathrm{ALG}(\mathbf{v})]$

   $$\begin{aligned}
   &= (1-\chi)\,\hat{v} + \mathbf{E}[\text{amount selected prize exceeds } \hat{v}] \\
   &\geq (1-\chi)\,\hat{v} + \sum_i \mathbf{E}[(v_i - \hat{v})^+ \,|\, \mathcal{E}_i]\,\mathbf{P}r[\mathcal{E}_i] \\
   &\geq (1-\chi)\,\hat{v} + \chi \sum_i \mathbf{E}[(v_i - \hat{v})^+]
   \end{aligned}$$

3. choose $\chi = 1/2$ and combine.

   $\mathbf{E}[\mathrm{ALG}(\mathbf{v})] \geq \frac{1}{2}\mathbf{E}[\mathrm{OPT}(\mathbf{v})]$

Exercise: Gambler's Threshold

Setup:

- n=2 buyers
- values $U[0,1]$

Question: What is the threshold of the prophet inequality, i.e., where probability of no prize equal to 1/2.

The Secretary Problem

"online allocation with 'buyers' in uniform random order"

Assumption 2: values are in a uniformly random order.

Example: $n = 3$

Two algs for example:

(a) give item to buyer i for some i

$\Rightarrow$ **Pr**[success] $= 1/3$

(b) look at 1st, condition choice of 2nd or 3rd.

- if 2nd greater than 1st, give to 2nd
- else, give to 3rd.

$\Rightarrow$ **Pr**[success] $= 1/2$

sequence	1 2 3	1 3 2	2 1 3	2 3 1	3 1 2	3 2 1
alg (a)					$\surd$	$\surd$
alg (b)		$\surd$	$\surd$	$\surd$		

Algorithm: Secretary Alg

1. solicit bids from k buyers, but do not sell.
2. give item to first remaining buyer with value greater than any of first k.

Lemma: For $k = n/2$ alg is selects highest value w.p. 1/4.

Proof:

- select highest when 2nd best in first half and 1st best in second half.
- Recall "conditional probability":

  $\mathbf{Pr}[A \,\&\, B] = \mathbf{Pr}[A \mid B]\mathbf{Pr}[B]$.
- **Pr**[2nd best in first half] $= 1/2$
- **Pr**[1st best in second half | 2nd best in first half] $= \frac{n/2}{n-1} \geq 1/2$

$\Rightarrow$ **Pr**[select highest]

$\geq$ **Pr**[2nd in 1st 1/2]**Pr**[1st in 2nd 1/2 | 2nd in 1st 1/2] $\geq 1/4$.

Q: what is best k?

Thm: for $k = n/e$ alg selects highest buyer with probability $1/e$

(and this is best possible; recall $e = 2.718$)