June 1

Monday, June 1, 2020 2:47 PM

CompSci 212

1. Coloring
2. Bipartite graphs

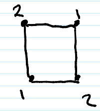
Announcements
- Homework 6 - due June 3
- Final, stay tuned

Coloring

Def. A graph $G=(V,E)$ is k-colorable if there exists a function $f: V \to [k]$ s.t. if $\{u,v\}$, then $f(u) \neq f(v)$.

$f(u)$: coloring; $[k]$: colors

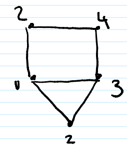
Def. The chromatic number $\chi(G)$ of a graph is the smallest k s.t. G is k-colorable.



2-colorable

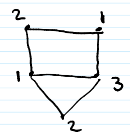
$\chi(G) \leq 2$

$\chi(G) = 2$



4-colorable

$\chi(G) \leq 4$



3-colorable

$\chi(G) \leq 3$

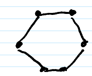
$\chi(G) = 3$

If Graph is k-colorable, $\chi(G) \leq k$.

$\chi(G) \leq |V|$, can assign every vertex a diff. color

If G has edges, $\chi(G) \geq 2$, need at least 2 colors for the two vertices in any edge.

Cycle $V=[n]$ $E=\{\{1,2\},\{2,3\},\ldots,\{n-1,n\},\{1,n\}\}$



Thm. If $G=([n],E)$ is a cycle w/ n vertices, and n is even, $\chi(G)=2$.

Proof. G has edges, $\chi(G) \geq 2$

Let $f: [n] \to [2]$ be

$f(x) = \begin{cases} 1 & \text{if } v \text{ is odd} \\ 2 & \text{if } v \text{ is even.} \end{cases}$

Then $f(i) \neq f(i+1)$ for all i, $f(1) \neq f(n)$, f is a valid 2-coloring. □

(with $f(1) = 1$, $f(n) = 2$)

Thm. If $G=([n],E)$ is a cycle and n is odd, $\chi(G)=3$. $(*)$

Proof. G has edges, $\chi(G) \geq 2$.

Prove $\chi(G) > 2$ by contradiction.

Assume that $f: V \to \{1,2\}$ is a valid 2-coloring. Then $f(1) \neq f(2)$, $f(2) \neq f(3)$, therefore $f(1)=f(3)$.

Similarly, $f(3)=f(5)$ and in general $f(1)=f(3)=f(5)=\cdots=f(n)$, but $\{1,n\} \in E$, contradiction, f is not a valid coloring.

G is 3-colorable, $f(v) = \begin{cases} 1 & \text{if } v \text{ is odd, less than } n \\ 2 & \text{if } v \text{ is even} \\ 3 & \text{if } v \text{ is } n \end{cases}$

$f(i) \neq f(i+1)$ for all i, $f(1) \neq f(n)$ (with $f(1)=1$, $f(n)=3$)

so f is a valid coloring. □

Complete graph, $V=[n]$, $E=$ all possible edges.

Thm. If $G=([n],E)$ is the complete graph, $\chi(G)=n$.

Proof. All colors must be different from each other.

$\chi(G) \leq |V|$ for all G

($f: [n] \to [n]$ $f(x)=x$ is a valid coloring).

Thm. If $G=(V,E)$ is a graph where all vertices have degree k or less, G is $(k+1)$-colorable.

Proof. Use induction on the # of vertices.

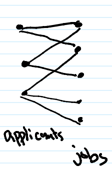
Base case: 1 vertex, obvious

Inductive step: Assume statement is true for $|V|-1$ vertices. Let $v \in V$, let $S=\{e \mid e \in E, v \in e\}$ (set of edges containing v)

Let $G'=(V\setminus\{v\}, E\setminus S)$, by inductive hypothesis, it's $(k+1)$-colorable.

Color vertices in G the same, let v have any color diff. from a neighbor. This is possible b/c $\deg(v) \leq k$, $k+1$ possible colors. □

Def. If G is 2-colorable, G is bipartite



Split V into V_1 + V_2 s.t. every edge goes from V_1 to V_2.

Thm. $G=(V,E)$ is 2-colorable iff it has no odd cycles - (cycles w/ an odd # of vertices)

Proof. If G has an odd cycle, then it is not 2-colorable, Thm $(*)$.

If G has no odd cycles, it's 2-colorable.

Def. length of a path # of edges in path

Def $\text{dist}(v,w)$ = smallest length of a path from w to v.

Pick arbitrary $v \in V$.

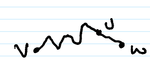
$f(w) = \begin{cases} 1 & \text{if } \text{dist}(v,w) \text{ is odd.} \\ 2 & \text{if } \text{dist}(v,w) \text{ is even} \end{cases}$

We claim f is a valid coloring.

Assume $f(u)=f(w)$, but $\{u,w\} \in E$.

$\text{dist}(v,u) = \text{dist}(v,w)+1$
$\text{dist}(v,w)-1$
$\text{dist}(v,w)$

(Assume $\text{dist}(v,u) \leq \text{dist}(v,w)$)



Additionally, $\text{dist}(v,u)$ + $\text{dist}(v,w)$ must have the same parity. This implies $\text{dist}(v,u)=\text{dist}(v,w)$.

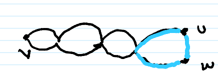
Let $(v, u_1, u_2, \ldots, u_\ell, u)$ path of length $\text{dist}(v,u)$

$(v, w_1, w_2, \ldots, w_\ell, w)$ path of length $\text{dist}(v,w)$

Let i be the largest s.t. $u_i = w_i$.

$(u_i, u_{i+1}, \ldots, u_\ell, u, w, w_\ell, \ldots, w_i)$

odd cycle (# of edges $2\cdot(\ell - i)+3$)



root. If G has an odd cycle, then
it is not 2-colorable, Thm (*).
If G has no odd cycles, it's 2-colorable.

Additionally, dist(v,u) & dist(v,w) must have the same
parity. This implies dist(v,u) = dist(v,w).