May 27

Wednesday, May 27, 2020 2:56 PM

Comp Sci 212

1. Trees

Announcements
- Final June 11

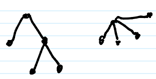
Def. A graph is acyclic if it contains no cycles.

Thm. Every graph $G = (V, E)$ has at least $|V| - |E|$ connected components.

Corollary. Every graph that is connected has at least $|V| - 1$ edges.

Corollary. If a graph is acyclic and $|E| = |V| - 1$ then it is connected and acyclic.

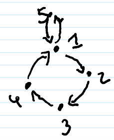
Trees -



Thm. The following are equivalent (and are all definitions of trees) $G = (V, E)$

1. G is connected & acyclic
2. Every pair of vertices is connected by a unique path
3. G is connected and $|E| = |V| - 1$.
4. G is acyclic and $|E| = |V| - 1$.
5. G is acyclic and adding any new edge creates a cycle.

Proof.



$2 \to 3$

G is connected

Prove that $|E| = |V| - 1$ using induction.

Base case $|V| = 1$, obvious

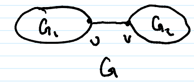
Inductive step, assume true $|V| \le k$.

If $|V| = k+1$ vertices. Let $\{u, v\} \in E$

$G' = (V, E \setminus \{\{u, v\}\})$

In G', u & v must be disconnected, or else there would be 2 paths from u to v in G.

Two connected components in G', G_1 & G_2



G_1, G_2 together make up G'

$G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$, G_1 & G_2 satisfy 2 therefore $|E_i| = |V_i| - 1$

$E = E_1 \cup E_2 \cup \{\{u, v\}\}$ $\quad V = V_1 \cup V_2$

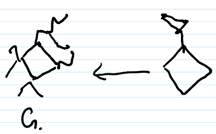
$|E| = |V_1| - 1 + |V_2| - 1 + 1 = |V| - 1$

$3 \to 4$ If G is connected and $|E| = |V| - 1$, then G is acyclic.

Use contradiction.

Assume G has a cycle, k vertices, k edges.

Start with the cycle, and add vertices one by one until we G.



Adding each new vertex must involve adding a new edge. At the end, $|E| \ge |V|$, contradicts assumption that $|E| = |V| - 1$

$\square$

$4 \to 1$ in class Friday

1. G is connected & acyclic
5. G is acyclic & adding any new edge creates a cycle.

$5 \to 1$

Use contradiction - assume G is not connected. In particular assume u & v are not connected. Adding $\{u, v\}$ to E will not create a cycle. Contradicts the assumption in 5.

$1 \to 5$

Let $\{u, v\}$ be a new edge. Because u & v are connected, there is a path from u to v, $(u, v_1, v_2, \ldots, v_n, v)$, adding $\{u, v\}$ turn this into a cycle.

$$R = \sum_{i=1}^{n-1} \mathbb{1}_{E_i}$$

$$\mathbb{E}[R^2] = \mathbb{E}\left[\left(\sum_{i=1}^{n-1} \mathbb{1}_{E_i}\right)^2\right] = \sum_{i,j=1}^{n-1} \mathbb{E}[\mathbb{1}_{E_i} \mathbb{1}_{E_j}]$$

(coins Chebyshev)

$$= \sum_{i=j} \mathbb{E}[\mathbb{1}_{E_i} \mathbb{1}_{E_j}]$$

$(n-1)^2$ terms total

$$+ \sum_{|i-j|=1} \mathbb{E}[\mathbb{1}_{E_i} \mathbb{1}_{E_j}]$$

# of terms 1st sum + # of terms 2nd sum + # terms 3rd sum $= (n-1)^2$

$$+ \sum_{|i-j| \ge 2} \mathbb{E}[\mathbb{1}_{E_i} \mathbb{1}_{E_j}]$$