May 22

Friday, May 22, 2020 2:33 PM

CompSci 212

1. Connected components
2. Trees

Graph - $G=(V, E)$, $E \subseteq V \times V$

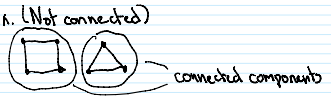
(unordered → pairs → subsets)

Path - sequence of vertices $(v_1, \dots, v_n)$ s.t. $\{v_i, v_{i+1}\} \in E$ for all i (repeats okay)

u & v are connected if there exists a path from u to v.

A graph is connected if every pair of vertices is connected.

Ex. (Not connected)



connected components

Def. subgraph of $G=(V, E)$ is the graph $G'=(V', E')$ s.t. $V' \subseteq V$, $E' \subseteq E$ if $\{u,v\} \in E'$, then $u, v \in V'$.

Def. connected component - a subgraph of G containing a vertex v, all vertices connected to v, all edges between such vertices.

- (a connected graph has just 1 connected component)

Thm. Every graph $G=(V, E)$ has at least $|V| - |E|$ connected components.

Proof. Use induction on the number of edges.

Base case - $|E|=0$, there are $|V|$ cc's.

Inductive step - Assume statement is true if $|E|=k$, prove the statement for $|E|=k+1$.

Let $\{u,v\} \in E$, consider graph $G'=(V, E \setminus \{u,v\})$

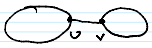
By inductive hypothesis, G' has at least $|V| - k$ cc's.

Either u & v are in the same connected component in G' (Case 1), or different (Case 2)

Case 1 - adding edge $\{u,v\}$ keeps # of cc's the same

# of cc's in G = # of cc's in $G' \geq |V| - k \geq |V| - (k+1) = |V| - |E|$

Case 2



adding edge $\{u,v\}$ decreases # of cc's by 1

Therefore,

# of cc's in G = # of cc's in G' - 1

$\geq |V| - k - 1 = |V| - |E|$ □

Def (acyclic) a graph is acyclic if no subgraph is a cycle. (⬠)

Corollary. If a graph is acyclic it has exactly $|V| - |E|$ connected components.

Proof. Need to show never in Case 1. (if never in Case 1, can do same proof but w/ equality instead of inequality).

Proof by contradiction. Assume that $\{u,v\}$ already connected before adding $\{u,v\}$ as an edge.

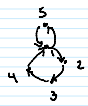
There was a path from u to v, $(u, v_1, v_2, v_3, \dots, v)$, adding $\{u,v\}$ creates a cycle, contradicts the assumption that G is acyclic.

Corollary. If a graph is acyclic and has $|E| = |V| - 1$ logs, then it has 1 connected component, it's connected & acyclic.

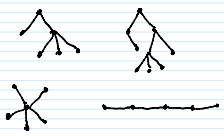
Thm. The following are equivalent (definitions of trees.)

1. G is connected and acyclic
2. Every pair of vertices is connected by a unique path.
3. G is connected and $|E| = |V| - 1$.
4. G is acyclic and $|E| = |V| - 1$.
5. G is acyclic, adding any new edge creates a cycle.

Proof. Show that $1 \to 2$, $2 \to 3$, $3 \to 4$, $4 \to 1$, $5 \to 1$, $1 \to 5$



Trees



$1 \to 2$ Use contradiction.

If there exist two vertices, u & v not connected, then G is not connected.

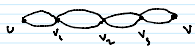
Assume there exist two vertices, u & v, connected by more than one path. Let

$P_1 = (u, x_1, x_2, x_3, \dots, x_m, v)$

$P_2 = (u, y_1, y_2, \dots, y_n, v)$

be two paths.

Let $(u, v_1, v_2, \dots, v_i, v)$ be the vertices that the two paths P_1 & P_2 have in common.



Let j be s.t. v_j, v_{j+1} are not subsequent in both P_1 & P_2. Such j must exist because paths different.

$v_j = x_a, y_c$

$v_{j+1} = x_b, y_d$ (y_d = y_{d-1})

Then $(x_a, x_{a+1}, \dots, x_b, y_{d-1}, \dots, y_c)$ is a cycle.