May 15

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Comp Sci 212

1. Markov's Inequality

2. Chebyshev's Inequality

Announcements

- Midterm grades out
- Getting to know you meetings

Tails bounds

R random variable

$\Pr[R \geq x] \leq$?

- If R represent the running time of a random algorithm, (or the error), want $\Pr[R \geq x]$ to be small.

Markov's Inequality

What tails bounds bounds can you obtain from just the expected value?

Ex. In a room of 100 people, avg. age 40

S = set of people

$\Pr[\omega] = \frac{1}{100}$ for each $\omega \in S$

$R: S \to \mathbb{R}$, $R(\omega)$ age of person ω

$\mathbb{E}[R] = 40 = \sum_{\omega \in S} R(\omega) \cdot \Pr[\omega]$

What is the largest $\Pr[R \geq 100]$ can be? proportion of people ≥ 100 years.

Not everyone can be at least 100

Even $\Pr[R \geq 100] = \frac{1}{2}$ is impossible

worst case 50 people 100

50 people 0, $\to$ avg. 50

$\Pr[R \geq 100] = \frac{1}{4}$ is possible

25 people be 100

75 people be 20 $\to$ avg. 40

largest is $\Pr[R \geq 100] = \frac{4}{10} = \frac{\mathbb{E}[R]}{100}$

40 people 100

60 people 0 $\to$ avg. 40

make everyone either 100 or 0.

Markov's Inequality

If R is a non-negative random variable $x > 0$, then

$$\Pr[R \geq x] \leq \frac{\mathbb{E}[R]}{x}$$

Proof. Enough to prove

$\mathbb{E}[R] \geq x \Pr[R \geq x]$

$\sum_{\omega \in S} R(\omega) \Pr[\omega]$ $\qquad$ $\sum_{\substack{\omega \in S \\ \text{s.t. } R(\omega) \geq x}} \Pr[\omega] \cdot x$

$$\sum_{\omega \in S} R(\omega)\Pr[\omega] = \sum_{\substack{\omega \in S \\ \text{s.t. } R(\omega) \geq x}} R(\omega)\Pr[\omega] + \underbrace{\sum_{\substack{\omega \in S \text{ s.t.} \\ R(\omega) < x}} R(\omega)\Pr[\omega]}_{\text{at least } 0}$$

$$\geq \sum_{\substack{\omega \in S \text{ s.t.} \\ R(\omega) \geq x}} R(\omega)\Pr[\omega]$$

$$\geq \sum_{\substack{\omega \in S \text{ s.t.} \\ R(\omega) \geq x}} x \Pr[\omega] \qquad (R(\omega) \geq x)$$

$\square$

Coins - $S = \{H, T\}^{100}$

$R: S \to \mathbb{R}$, $R(\omega)$ = # of heads in ω

Uniform dist., sticky, $\mathbb{E}[R] = 50$

$\Pr[R \geq 75] \leq \frac{\mathbb{E}[R]}{75} \leq \frac{2}{3}$

(by Markov's inequality)

Sticky coin - $\Pr[R \geq 75] = \frac{1}{2}$

all heads up $\frac{1}{2}$

all tails up $\frac{1}{2}$

Uniform dist. $\Pr[R \geq 75]$ = really small

(example Markov's inequality doesn't give) a good bound

Birthday Paradox - D = set of days, $|D| = 365$

$S = D^n$ (sample space), uniform dist.

$R: D^n \to \mathbb{R}$, $R(\omega)$ = # of pairs of people w/ same birthday

$E_{i,j}$ = event that person i & person j have same birthday

$\Pr[E_{i,j}] = \frac{1}{365}$

$R = \sum_{\text{pairs } i,j} \mathbb{1}_{E_{i,j}} \to \mathbb{E}[R] = \sum_{\text{pairs } i,j} \Pr[E_{i,j}]$

$= \frac{\text{\# of pairs}}{365} = \frac{n(n-1)}{730}$

If $n \approx 27$, $\mathbb{E}[R] \approx 1$

$\Pr[R \geq 10] \leq \frac{\mathbb{E}[R]}{10} \approx \frac{1}{10}$

(by Markov's inequality)

Back to room (avg. age 40)

- Assume that everyone is at least 20 years old

40 people above 100 is now impossible

40 people 100

60 people 20 $\quad$ avg. 52

largest possible is $\Pr[R \geq 100] \leq \frac{25}{100}$

25 people 100 years

75 people 20 years

$\mathbb{E}[R] = 40$, $b = 20$ $\qquad = \frac{40 - 20}{100 - 20} = \frac{20}{80} = \frac{1}{4}$

Corollary. If $R(\omega) \geq b$ for all ω, $x > b$

$$\Pr[R \geq x] \leq \frac{\mathbb{E}[R] - b}{x - b}$$

($b = 0$, just Markov's inequality)

Proof. Let $Y(\omega) = R(\omega) - b$

$Y(\omega)$ is always non-negative

$\mathbb{E}[Y] = \mathbb{E}[R] - b$

$$\Pr[R \geq x] = \Pr[Y \geq x - b] \leq \frac{\mathbb{E}[Y]}{x - b} = \frac{\mathbb{E}[R] - b}{x - b}$$

(Markov's inequality)

$\square$