May 13

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Comp Sci 212

1. Binomial Distribution

2. Expectation of Products

Announcements

- Homework 4 out

Let $E_1, \dots, E_n$ events $R: S \to \mathbb{R}$

$R(s) = $ # of events that s is in

$\mathbb{E}[R] = \sum_{i=1}^{n} \Pr[E_i]$

Binomial distribution ($B(n, p)$) — p anything between 0, 1

Sample space $S = \{0,1\}^n$

$\Pr[\omega] = p^{\#\text{ of 1's in }\omega}(1-p)^{\#\text{ of 0's in }\omega}$

$\sum_{\omega \in \{0,1\}^n} \Pr[\omega] = \sum_{i=0}^{n} (\#\text{ of strings w/ } i \text{ 1's}) p^i (1-p)^{n-i}$

$= \sum_{i=0}^{n} \binom{n}{i} p^i (1-p)^{n-i}$

$= (p + (1-p))^n$ (by binomial thm)

$= 1$

$\sum_{\omega \in \{0,1\}^n} \Pr[\omega] = \sum_{\omega \in \{0,1\}^n} p^{\#\text{ of 1's in }\omega}(1-p)^{\#\text{ of 0's in }\omega}$

$= (p + (1-p))^n = 1$

$R: S \to \mathbb{R}$, $R(\omega) = $ # of 1's in ω

Compute $\mathbb{E}[R]$?

$\mathbb{E}[R] = \sum_{\omega \in S} \Pr[\omega](\#\text{ of 1's in } s)$

$= \sum_{i=0}^{n} i\,(\#\text{ of strings w/ } i \text{ 1's}) p^i (1-p)^{n-i}$

$= \sum_{i=0}^{n} i \binom{n}{i} p^i (1-p)^{n-i}$ – ???

$= pn$

- If $p = \frac{1}{2}$, uniform dist. $\Pr[\omega] = \frac{1}{2^n}$ for all ω

- If $p = \frac{3}{4}$, $n = 5$, $\omega = (1,1,1,0,0)$

$\Pr[\omega] = \left(\frac{3}{4}\right)^3\left(1-\frac{3}{4}\right)^2 = \left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)^2$

$E_i = \{\omega \mid \omega \in \{0,1\}^n, \omega_i = 1\}$

event where i^{th} coordinate is 1

$R(\omega) = $ # of 1's in ω

$=$ # of events ω is in

$\mathbb{E}[R] = \sum_{i=1}^{n} \Pr[E_i] = \sum_{i=1}^{n} p = pn$

$\Pr[E_i] = \sum_{\substack{\omega \in \{0,1\}^n \\ \text{s.t. } \omega_i = 1}} \Pr[\omega]$

$= \sum_{\substack{\omega \in \{0,1\}^n \\ \text{s.t. } \omega_i = 1}} p^{\#\text{ of 1's in }\omega}(1-p)^{\#\text{ of 0's in }\omega}$

$= p \underbrace{\sum_{\omega \in \{0,1\}^{n-1}} p^{\#\text{ of 1's in }\omega}(1-p)^{\#\text{ of 0's in }\omega}}_{=1,\ \text{distribution } B(n-1,\,p)}$

$= p$

Fact. $\mathbb{E}[X] = \sum_{r \in \text{Range}(X)} r \Pr[X = r]$

$\text{Range}(X)$: set of all values X takes on

$\text{Range}(X) = \{r \mid X(\omega) = r \text{ for some } \omega \in S\}$

Proof. $\mathbb{E}[X] = \sum_{\omega \in S} X(\omega)\Pr[\omega]$

$= \sum_{r \in \text{Range}(X)} \sum_{\omega:\, X(\omega) = r} \Pr[\omega] X(\omega)$

$= \sum_{r \in \text{Range}(X)} r \sum_{\omega:\, X(\omega) = r} \Pr[\omega]$

$= \sum_{r \in \text{Range}(X)} r \Pr[X = r]$ □

Expected Value of Products

Thm. If R_1, R_2 are independent r.v.'s → (def. independence for random variables) for all r_1, r_2, $\Pr[R_1 = r_1 \text{ and } R_2 = r_2] = \Pr[R_1 = r_1] \cdot \Pr[R_2 = r_2]$

$\mathbb{E}[R_1 R_2] = \mathbb{E}[R_1]\mathbb{E}[R_2]$

$R_1 R_2 : S \to \mathbb{R}$

$R_1 R_2(\omega) = R_1(\omega) \cdot R_2(\omega)$

Proof. $\mathbb{E}[R_1 R_2] = \sum_{\substack{r_1 \in \text{Range}(R_1) \\ r_2 \in \text{Range}(R_2)}} \sum_{\substack{\omega:\, R_1(\omega) = r_1 \\ R_2(\omega) = r_2}} r_1 \cdot r_2 \cdot \Pr[\omega]$ (this line changed slightly from recording)

$= \sum_{\substack{r_1 \in \text{Range}(R_1) \\ r_2 \in \text{Range}(R_2)}} \Pr[R_1 = r_1 \text{ and } R_2 = r_2] \cdot r_1 \cdot r_2$

$= \sum_{\substack{r_1 \in \text{Range}(R_1) \\ r_2 \in \text{Range}(R_2)}} \Pr[R_1 = r_1] \cdot \Pr[R_2 = r_2] \cdot r_1 \cdot r_2$

$= \left(\sum_{r_1 \in \text{Range}(R_1)} \Pr[R_1 = r_1] \cdot r_1\right)\left(\sum_{r_2 \in \text{Range}(R_2)} \Pr[R_2 = r_2] \cdot r_2\right) = \mathbb{E}[R_1] \cdot \mathbb{E}[R_2]$ □

Cauchy-Schwartz

Thm. If X, Y are real-valued r.v.'s

$\mathbb{E}[XY]^2 \le \mathbb{E}[X^2]\mathbb{E}[Y^2]$

$(A_{\omega_1} + A_{\omega_2} + A_{\omega_3} + \dots)(A_{\omega_1} + A_{\omega_2} + A_{\omega_3} + \dots)$

Proof. $\mathbb{E}[XY]^2 = \left(\sum_{\omega \in S} \Pr[\omega] X(\omega) Y(\omega)\right)^2$

$= \sum_{\omega \in S} \Pr[\omega]^2 X(\omega)^2 Y(\omega)^2$

$+ \sum_{\{\omega, v\} \subseteq S} 2\Pr[\omega]\Pr[v] X(\omega) Y(\omega) X(v) Y(v)$

$\mathbb{E}[X^2]\mathbb{E}[Y^2] = \left(\sum_{\omega \in S} \Pr[\omega] X(\omega)^2\right)\left(\sum_{\omega \in S} \Pr[\omega] Y(\omega)^2\right)$

$= \sum_{\omega \in S} \Pr[\omega]^2 X(\omega)^2 Y(\omega)^2 + \sum_{\{\omega, v\} \subseteq S} \Pr[\omega]\Pr[v]\left(X(\omega)^2 Y(v)^2 + X(v)^2 Y(\omega)^2\right)$

$\mathbb{E}[X^2]\mathbb{E}[Y^2] - \mathbb{E}[XY]^2 = \sum_{\{\omega, v\} \subseteq S} \Pr[\omega]\Pr[v]\left(X(\omega)^2 Y(v)^2 - 2X(\omega)Y(\omega)X(v)Y(v) + X(v)^2 Y(\omega)^2\right)$

$(X(\omega)Y(v) - X(v)Y(\omega))^2 \ge 0$ □