May 8

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Comp Sci 212

1. Expected Value
2. Linearity of Expectation

Announcements

- Midterm May 11

Expected Value

Def. (Expected Value) X is random variable, Sample space S, Probability function $\Pr$, $X: S \to A$ (usually $A = \mathbb{R}$)

$$\underset{\substack{\uparrow \\ \text{expected}}}{\mathbb{E}}[X] = \sum_{w \in S} X(w) \Pr[w]$$

(like mean, average)

Ex 6-sided die $S = \{1, 2, 3, 4, 5, 6\}$

$X: S \to \mathbb{R}$ $X(w) = w$

uniform dist., $\Pr[w] = \frac{1}{6}$ for all w,

$$\mathbb{E}[X] = \sum_{w \in S} X(w) \cdot \Pr[w] = \frac{1}{6} + \frac{2}{6} + \frac{3}{6} + \frac{4}{6} + \frac{5}{6} + \frac{6}{6} = \frac{7}{2}$$

$\Pr[1] = \frac{1}{2}$, $\Pr[w] = \frac{1}{10}$ for $w \geq 2$

$$\mathbb{E}[X] = \sum_{w \in S} X(w) \cdot \Pr[w] = \frac{1}{2} \cdot 1 + \frac{2}{10} + \frac{3}{10} + \frac{4}{10} + \frac{5}{10} + \frac{6}{10} = \frac{5}{2}$$

Ex. Uniform on 6-sided die, $Y: S \to \mathbb{R}$,

$Y(w) = \frac{1}{w}$, $Y = \frac{1}{X}$

$$\mathbb{E}[Y] = \mathbb{E}\left[\frac{1}{X}\right] = \sum_{w \in S} Y(w) \cdot \Pr[w] =$$

$$\frac{1}{6} \cdot 1 + \frac{1}{6} \cdot \frac{1}{2} + \frac{1}{6} \cdot \frac{1}{3} + \frac{1}{6} \cdot \frac{1}{4} + \frac{1}{6} \cdot \frac{1}{5} + \frac{1}{6} \cdot \frac{1}{6} = \frac{49}{120}$$

$$\mathbb{E}[Y] \neq \frac{1}{\mathbb{E}[X]}$$

Def. Conditional Expectation, X r.v. (random variable), A event

$$\mathbb{E}[X \mid A] = \sum_{w \in A} X(w) \Pr[\{w\} \mid A] = \frac{1}{\Pr[A]} \sum_{w \in A} X(w) \Pr[w]$$

Law of total expectation, R random variable, $A_1, A_2, \dots, A_n$ disjoint events, $A_1 \cup A_2 \cup \dots \cup A_n = S$ (sample space)

$$\mathbb{E}[R] = \sum_{i=1}^{n} \mathbb{E}[R \mid A_i] \cdot \Pr[A_i]$$

Mean time until failure

Each day, your computer survives w.p. $(1-p)$, or fails w.p. p, how long will your computer last on average?

$S = \{1, 2, 3, \dots\}$

$\Pr[i] = (1-p)^{i-1} \cdot p$

$$\sum_{i=1}^{\infty} \Pr[i] = \sum_{i=1}^{\infty} p(1-p)^{i-1} = p \sum_{i=1}^{\infty} (1-p)^{i-1} = p \cdot \frac{1}{1-(1-p)} = 1$$

(formula for geometric series)

$R: S \to \mathbb{R}$, $R(w) = w$

$$\mathbb{E}[R] = \sum_{i=1}^{\infty} R(i) \Pr[i] = \underline{\sum_{i=1}^{\infty} p(1-p)^{i-1} \cdot i}$$

(can compute)

A_1, A_2, $A_1 = \{1\}$, $A_2 = S \setminus A_1 = \{2, 3, \dots\}$

$\Pr[A_1] = \Pr[1] = p$, $\Pr[A_2] = 1-p$

$$\mathbb{E}[R] = \mathbb{E}[R \mid A_1] \Pr[A_1] + \underline{\mathbb{E}[R \mid A_2] \Pr[A_2]}$$

(by law of total expectation)

$$\mathbb{E}[R \mid A_2] = \frac{1}{1-p} \sum_{i=2}^{\infty} i(1-p)^{i-1} p = \sum_{i=2}^{\infty} i(1-p)^{i-2} \cdot p$$

$$= \sum_{i=1}^{\infty} (i+1)(1-p)^{i-1} \cdot p$$

$$= \sum_{i=1}^{\infty} (1-p)^{i-1} p + \sum_{i=1}^{\infty} i(1-p)^{i-1} p$$

$$= 1 + \mathbb{E}[R]$$

$$\mathbb{E}[R] = 1 \cdot p + (1 + \mathbb{E}[R])(1-p)$$

$$= 1 + (1-p)\mathbb{E}[R]$$

$$\underline{\mathbb{E}[R] = \frac{1}{p}}$$

Linearity of Expectation.

Thm. $\mathbb{E}[R_1 + R_2] = \mathbb{E}[R_1] + \mathbb{E}[R_2]$

$R_1 + R_2: S \to A$, $R_1 + R_2(w) = R_1(w) + R_2(w)$

Proof.
$$\mathbb{E}[R_1 + R_2] = \sum_{w \in S} R_1 + R_2(w) \Pr[w]$$
$$= \sum_{w \in S} R_1(w) \Pr[w] + R_2(w) \Pr[w]$$
$$= \sum_{w \in S} R_1(w) \Pr[w] + \sum_{w \in S} R_2(w) \Pr[w]$$
$$= \mathbb{E}[R_1] + \mathbb{E}[R_2] \qquad \square$$

Def. (Indicator r.v.) Random variable whose range $\{0, 1\}$, usually associated w/ event

$$\mathbf{1}_E : S \to \{0, 1\} \qquad \mathbf{1}_E(w) = \begin{cases} 1 & \text{if } w \in E \\ 0 & \text{o.w.} \end{cases}$$

Ex. $S = \{H, T\}^{100}$

E_i = event that i^{th} coin is heads

$R: S \to \mathbb{R}$, $R(w)$ = # of heads in w

$R(w) = \sum_{i=1}^{100} \mathbf{1}_{E_i}(w)$ (both uniform, sticky coin)

$$\mathbb{E}[R] = \sum_{i=1}^{100} \mathbb{E}[\mathbf{1}_{E_i}] = \sum_{i=1}^{100} \Pr[E_i] = \sum_{i=1}^{100} \frac{1}{2} = 50$$

(by linearity of expectation)

$$\mathbb{E}[\mathbf{1}_E] = \sum_{w \in S} \mathbf{1}_E(w) \Pr[w]$$
$$= \sum_{w \in E} \Pr[w] = \Pr[E]$$

(all indicator r.v.'s)

Generally, $E_1, \dots, E_n$, $R: S \to \mathbb{R}$, $R(w)$ = # of events w is in

$$\mathbb{E}[R] = \sum_{i=1}^{n} \Pr[E_i]$$