May 4

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Comp Sci 212

1. Conditional Probability

2. Independence

Announcements

- Midterm May 11

- Collaboration Polic

Def. $Pr[A|B] = \frac{Pr[A\cap B]}{Pr[B]}$

Law of total probability

Same sample space

Pr, probability function

Thm If $E_1, \ldots, E_n$ are disjoint

$\sum_{i=1}^{n} Pr[A|E_i]Pr[E_i] =$

$Pr[A\cap(E_1 \cup E_2 \cup \cdots \cup E_n)]$

Useful if equal to sample space

Proof $Pr[A|E_i]\cdot Pr[E_i] = \sum_{i=1}^{n} Pr[A\cap E_i]$ (by def)

$= \sum_{i=1}^{n}\sum_{w\in A\cap E_i} Pr[w]$

Sum rule, because $A\cap E_i$'s are disjoint

Because E_i are disjoint, each w appears at most once

$= \sum_{w\in A\cap(E_1\cup E_2\cup\cdots\cup E_n)} Pr[w] = Pr[A\cap(E_1\cup E_2\cup\cdots\cup E_n)]$

Ex. Roll two dice, what is the probability that the sum is 4?

$S = [6]\times[6]$, uniform dist.

E_i = event 1st roll is i

A = event sum is 4

$Pr[A]$?

$E_1\cup E_2\cup\cdots\cup E_6 = S$

$Pr[A\cap(E_1\cup E_2\cup\cdots\cup E_6)]$

$= \sum_{i=1}^{6} Pr[A|E_i]\cdot Pr[E_i]$

$= \sum_{i=1}^{3}\frac{1}{6}\cdot\frac{1}{6} + \sum_{i=4}^{6} 0\cdot\frac{1}{6}$

$= \frac{1}{12}$

$\rightarrow Pr[A|E_i] = \frac{Pr[A\cap E_i]}{Pr[E_i]}$

$|A\cap E_i| = 1$ if $i\le 3$, 0 o.w.

$|E_i| = 6$ for all i

$= \begin{cases}\frac{1}{6} & \text{if } i\le 3\\ 0 & \text{o.w.}\end{cases}$

$Pr[E_i] = \frac{6}{|S|} = \frac{6}{36} = \frac{1}{6}$

Independence -

Def. Events A and B are independent if $Pr[A\cap B] = Pr[A]\cdot Pr[B]$

If $Pr[B]\ne 0$, $Pr[A|B] = Pr[A]$

$E_1,\ldots,E_n$ are mutually independent if for every subset S of $[n]$

$Pr[\bigcap_{j\in S} E_j] = \prod_{j\in S} Pr[E_j]$

$\rightarrow$ take intersection of E_j, but only the j in S

(Often independence is something you need to prove)

$C = \{H,T\}^{100}$ (sample space), uniform dist.

E_i = event ith coin is heads — subset of sequences w/ ith coin heads

$Pr[E_i] = \frac{|E_i|}{2^{100}} = \frac{2^{99}}{2^{100}} = \frac{1}{2}$ (generalized product rule)

$Pr[\bigcap_{j\in S} E_j] = \frac{|\bigcap_{j\in S} E_j|}{2^{100}} = \frac{\text{\# of ways to flip coins s.t. } j\text{th coin is heads for all } j\in S}{2^{100}}$

$S\subseteq[100]$

$= \frac{2^{100-|S|}}{2^{100}}$ (generalized product rule)

$= \frac{1}{2^{|S|}}$

$= \prod_{j\in S}\frac{1}{2}$ — multiply by $\frac{1}{2}$ for each j in S

$= \prod_{j\in S} Pr[E_j]$

Sticky coins

$Pr[\text{all heads}] = \frac{1}{2}$

$Pr[\text{all tails}] = \frac{1}{2}$

$Pr[\text{anything else}] = 0$

$Pr[E_i] = \frac{1}{2}$

$Pr[\bigcap_{j\in S} E_j] = \frac{1}{2}$, for all non-empty S

$= \sum_{w\in\bigcap_{j\in S} E_j} Pr[w] = Pr[\text{all heads}] = \frac{1}{2}$

not indepent

Def. Pairwise independence

$E_1,\ldots,E_n$ are pairwise ind. if for every i,j

$Pr[E_i\cap E_j] = Pr[E_i]\cdot Pr[E_j]$

Ex $C = \{H,T\}^2$ (flipping 2 coins) uniform dist.

E_1 = 1st coin heads {HH, HT}

E_2 = 2nd coin heads {TH, HH}

E_3 = both coins same {HH, TT}

$Pr[E_1\cap E_2] = \frac{1}{4}$

$Pr[E_1\cap E_3] = \frac{1}{4}$

$Pr[E_2\cap E_3] = \frac{1}{4}$

$Pr[E_i] = \frac{1}{2}$ for all i

{HH}

$Pr[E_1\cap E_2\cap E_3] = \frac{1}{4} \ne (\frac{1}{2})^3$

E_1, E_2, E_3 are pairwise ind. not mutually ind.