May 1

Friday, May 1, 2020 2:50 PM

Comp Sci 212

1. Probability
2. Conditional Probability

Announcements

- Getting to know you meetings
- Practice midterms

Probability

Def. Sample space - non-empty set S

$w \in S$, element of S, outcome

$E \subseteq S$, subset of S, event

Ex. $\{1, 2, 3, 4, 5, 6\}$

$\{\text{Heads}, \text{Tails}\}$

Def. Probability function - total function

$Pr: S \to \mathbb{R}$ s.t.

- $Pr[w] \geq 0$ for all $w \in S$
- $\sum_{w \in S} Pr[w] = 1$

(sum over all $w \in S$)

Ex. $S = \{1, 2, 3, 4, 5, 6\}$

w	$Pr[w]$
1	$\frac{1}{6}$
2	$\frac{1}{6}$
3	$\frac{1}{6}$
4	$\frac{1}{6}$
5	$\frac{1}{6}$
6	$\frac{1}{6}$

w	$Pr[w]$
1	$\frac{1}{2}$
2	$\frac{1}{10}$
3	$\frac{1}{10}$
4	$\frac{1}{10}$
5	$\frac{1}{10}$
6	$\frac{1}{10}$

If $E \subseteq S$, (event), $Pr[E] = \sum_{w \in E} Pr[w]$

Ex. $E = \{1, 2, 3\}$

$Pr[E] = Pr[1] + Pr[2] + Pr[3]$

$Pr[X \leq 3]$

$\{x \mid x \in S,\ x \leq 3\}$

Ex. Uniform distribution - $Pr[w] = \frac{1}{|S|}$

for all w

$Pr[E] = \frac{|E|}{|S|}$

$S'^2 = S' \times S' = \{HH, HT, TH, TT\}$

Let $S' = \{\text{Heads}, \text{Tails}\}$

$S = S'^{100} = \underbrace{S' \times S' \times \cdots \times S'}_{100 \text{ times}}$

flipping a coin 100 times

If we have uniform dist. $\left(Pr[w] = \frac{1}{2^{100}}\right)$

$Pr[\text{At least 75 heads}] = \frac{|E|}{2^{100}}$

$Pr[E]$; $E = \{s \mid s \in S$, at least 75 coordinates in s are Heads$\}$

$|E| = \sum_{i=75}^{100} \binom{100}{i}$

Sticky coin

$Pr[\text{All heads}] = \frac{1}{2}$ $\quad Pr[\text{any other outcome}] = 0$

$Pr[\text{All tails}] = \frac{1}{2}$ $\quad Pr[\text{At least 75 heads}] = \frac{1}{2}$

Probability rules

- Sum rule, if $E_1, \ldots, E_n$ are disjoint,

$Pr[E_1 \cup E_2 \cup \cdots \cup E_n] = \sum_{i=1}^{n} Pr[E_i]$

- Complement

$Pr[S \setminus E] = 1 - Pr[E]$

- Difference

$Pr[A \setminus B] = Pr[A] - Pr[A \cap B]$

Principle of Inclusion Exclusion

$Pr[A \cup B] = Pr[A] + Pr[B] - Pr[A \cap B]$

If $A \subseteq B$, $Pr[A] \leq Pr[B]$

Union bound

$Pr[A \cup B] \leq Pr[A] + Pr[B]$

$Pr[A_1 \cup A_2 \cup \cdots A_n] \leq \sum_{i=1}^{n} Pr[A]$

equality only if the events are disjoint

Union bound - 100-sided die ($S' = [100]$, uniform dist)

throw it 25 times

$S = \underbrace{S' \times S' \times \cdots \times S'}_{25 \text{ times}}$, uniform dist.

$Pr[\text{At least 1 roll is 1}] = Pr[A_1 \cup A_2 \cup \cdots \cup A_{25}]$

$A_i = \{s \mid s \in S,\ s_i = 1\}$ $\quad \leq \sum_{i=1}^{25} Pr[A_i]$

↳ event i^{th} roll is 1

$= \sum_{i=1}^{25} \frac{|A_i|}{|S|} = \sum_{i=1}^{25} \frac{100^{24}}{100^{25}} = \sum_{i=1}^{25} \frac{1}{100} = \frac{1}{4}$

Conditional Probability

6-sided die, roll it twice, given that the sum is 4, what is the probability that the 1^{st} roll is 1? (uniform dist.)

B = set of outcomes w/ sum of 4

$= \{(3,1), (2,2), (1,3)\}$

A = set of outcomes w/ 1^{st} roll of 1

$= \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6)\}$

$Pr[A \mid B] \overset{\text{def}}{=} \frac{Pr[A \cap B]}{Pr[B]}$

(probability of A given B)

Ex. $A \cap B = \{(1,3)\}$ $\quad |A \cap B| = 1$

$|B| = 3$

$Pr[A \cap B] = \frac{1}{36}$ $\quad Pr[A \mid B] = \frac{1}{3}$

$Pr[B] = \frac{3}{36}$,

Given that a person has two kids, one is a boy, what is the probability that both are boys?

$Pr[A \mid B] = \frac{Pr[A \cap B]}{Pr[B]} = \frac{\frac{|A \cap B|}{4}}{\frac{3}{4}} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$

B = events w/ one boy $= \{(b,g), (g,b), (b,b)\}$

A = events w/ two boys $= \{(b,b)\}$

$S = \{(b,b), (b,g), (g,b), (g,g)\}$ uniform dist.