April 29
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Comp Sci 212
1. Pigeonhole Principle
2. More bijection rule

Announcements
- Office Hours - raise your hand or send a message to enter a breakout room.
- Getting to know you
- Homework submissions - let me know if something is not graded

Pigeonhole principle

If the number of pigeons is more than the number of holes, there exists at least one hole with more than one pigeon.

If $f: A \to B$, is total, and $k|B| < |A|$, then there exists a $b \in B$ s.t. $|f^{-1}(b)| \geq k+1$

Theorem. In a sock drawer w/ black + blue socks, picking out three socks will give you a pair w/ the same color.

Proof. Let $S = \{s_1, s_2, s_3\}$ (socks) $C = \{\text{black}, \text{blue}\}$ (colors)
$f: S \to C$, $f(s) =$ color of s.
$|S| > |C|$, by Pigeonhole principle, there exists $c \in C$ s.t. $|f^{-1}(c)| \geq 2 \to$ exists a pair w/ same color.

Theorem. Let $S \subseteq [20]$ ($\{1, 2, \dots, 20\}$) s.t. $|S| \geq 11$. Then there exist $i, j \in S$ $i - j = 10$.

Ex $S = \{\underline{1}, 2, 3, 4, 5, 6, 7, 8, 9, 10, \underline{11}\}$

Possible pairs - (1, 11) (2, 12), (3, 13), (4, 14) (5, 15) (6, 16), (7, 17), (8, 18) (9, 19) (10, 20)
pigeons

Proof. Let $f: S \to [10]$ be defined by
$$f(x) = \begin{cases} x - 10 & \text{if } x > 10 \\ x & \text{if } x \leq 10 \end{cases}$$
→ holes

$|S| \geq 11 > |[10]| = 10$, so by the Pigeonhole principle there exists at least one $b \in [10]$ s.t. $|f^{-1}(b)| \geq 2$, $i, j \in f^{-1}(b)$, $i > j$, then $i - j = 10$. ($i = b + 10$, $j = b$)

Thm. $\binom{n}{k} = \binom{n}{n-k}$ $\quad \binom{n}{k} = \frac{n!}{k!(n-k)!}$

Proof. $\binom{n}{k} = \#$ of subsets of $[n]$ of size k.
$= |A|$, $A =$ set of subsets of $[n]$ of size k
$\binom{n}{n-k} = \#$ of subsets of $[n]$ of size $n-k$
$= |B|$, $B =$ set of all subsets of $[n]$ of size $n-k$

Let $f: A \to B$,
$f(a) = [n] \setminus a$ (set of all elements in $[n]$ but not a) $\longrightarrow$ a, $f(a)$ are disjoint, $a \cup f(a) = [n]$

f is a bijection, $f^{-1}(b) = \{[n] \setminus b\}$, if $f(a) = b$, a, b are disjoint $a \cup b = [n]$
$|f^{-1}(b)| = 1$, (only one set yields b) □

Thm. $\binom{n}{k} = \sum_{i=0}^{k} \binom{n/2}{i} \binom{n/2}{k-i}$ (assume n is even)

Proof. $A =$ set of subsets of $[n]$ of size k
$|A| = \binom{n}{k}$
$B_i =$ set of subsets of $[n/2]$ of size i
$C_i =$ set of subsets of $[n] \setminus [n/2]$ of size $k-i$
($\{\frac{n}{2}+1, \frac{n}{2}+2, \dots, n\}$)

$$\text{RHS} = \sum_{i=0}^{k} |B_i||C_i| = \underbrace{\sum_{i=0}^{k} |B_i \times C_i|}_{\text{(product rule)}} = \underbrace{\left| \bigcup_{i=0}^{k} B_i \times C_i \right|}_{\text{(sum rule)}}$$
(union of all sets)

$D = \bigcup_{i=0}^{k} B_i \times C_i$

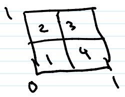
elements of D are of the form (b, c) where $b \subseteq [n/2]$ of size i for some i
$c \subseteq [n] \setminus [n/2]$ of size $k-i$ for same i □

$f: A \to D$, need to prove f is a bijection
$f(a) = (a \cap [n/2], a \cap ([n] \setminus [n/2]))$
$f^{-1}((b, c))$
If $f(a) = (b, c)$, then $a = b \cup c$
$f^{-1}(b, c) = \{b \cup c\}$,
$|f^{-1}(b, c)| = 1$, so f is a bijection

Theorem. 5 points in a 1×1 square, there exist at least 2 w/ distance less than $\frac{1}{\sqrt{2}}$

Proof. Use pigeonhole principle. 5 points will be pigeons, $= S$

set of boxes



$\frac{1}{2} \times \frac{1}{2}$

$f: S \to B$

$f(s) =$ box that s is in.

By pigeonhole principle, there exists $b \in B$ s.t. $|f^{-1}(b)| \geq 2$, there exists 2 points in one box, by geometry this implies their distance is $\leq \frac{1}{\sqrt{2}}$ $\square$