April 27

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Comp Sci 212

1. Binomial Theorem
2. Principle of Inclusion - Exclusion

Thm. Let S be a set of size k. Then, # of subsets of S of size ℓ is $\frac{k!}{\ell!(k-\ell)!} = \binom{k}{\ell}$ ("k choose ℓ")

Thm. $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$

Proof. Let A = set of subsets of $[n+1] = \{1, 2, \ldots n+1\}$ of size k.

Let A_1 = set of subsets of $[n]$ of size k

Let A_2 = set of subsets of $[n]$ of size $k-1$

$|A| = \binom{n+1}{k}$, $|A_1| = \binom{n}{k}$, $|A_2| = \binom{n}{k-1}$

$|A| = |A_1| + |A_2|$ (equivalent to thm)

A_1 & A_2 are disjoint, sum rule

$|A_1 \cup A_2| = |A_1| + |A_2|$, it's enough to prove $|A| = |A_1 \cup A_2|$

Use bijection rule, $f: A \to A_1 \cup A_2$

$$f(x) = \begin{cases} x & \text{if } x \text{ does not contain } n+1 \\ x \setminus \{n+1\} & \text{o.w.} \end{cases}$$

(If x does not contain $n+1$, $f(x) \in A_1$, if x does contain $n+1$, $f(x) \in A_2$)

To prove f is a bijection, need to show $f^{-1}(y) = \{x \mid f(x) = y\}$ has size $= 1$.

$$f^{-1}(y) = \begin{cases} y & \text{if } y \in A_1, \text{ or } |y| = k \\ y \cup \{n+1\} & \text{if } y \in A_2, \text{ or } |y| = k-1 \end{cases}$$

therefore $|f^{-1}(y)| = 1$, f is a bijection $\square$

(algebraic proof also works).

Def. $\binom{n}{k} = 0$ if $k \le -1$, or $k \ge n+1$

Notation. $\sum_{k=i}^{j} a_k = a_i + a_{i+1} + a_{i+2} \cdots a_j$

$\sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$

Binomial Theorem

Thm. If n is a non-negative integer, $(1+x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i$

Ex If $n=4$, $(1+x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4$

Proof. Use induction

Base case - $n = 0$, $1 = \binom{0}{0} x^0$ ✓

Inductive step - Assume, $(1+x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i$

Then, $(1+x)^{n+1} = (1+x)\underbrace{(1+x)^n}$

$= (1+x) \sum_{i=0}^{n} \binom{n}{i} x^i$ (by inductive hypothesis)

$= \sum_{i=0}^{n} \binom{n}{i} x^i + \sum_{i=0}^{n} \binom{n}{i} x^{i+1}$

$= \sum_{i=0}^{n} \binom{n}{i} x^i + \sum_{i=1}^{n+1} \binom{n}{i-1} x^i$

$\binom{n}{n+1} = 0$, $\binom{n}{-1} = 0$

$= \sum_{i=0}^{n+1} \binom{n}{i} x^i + \sum_{i=0}^{n+1} \binom{n}{i-1} x^i$

(two extra terms are $\binom{n}{n+1} x^{n+1}$, $\binom{n}{-1} x^0$, both 0)

$= \sum_{i=0}^{n+1} x^i \left(\binom{n}{i} + \binom{n}{i-1} \right)$

$= \sum_{i=0}^{n+1} x^i \binom{n+1}{i}$ by previous thm. $\square$

Corollary. If n is a non-negative integer, $(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$

Proof. $\left(1 + \frac{a}{b}\right)^n = \sum_{i=0}^{n} \binom{n}{i} \frac{a^i}{b^i}$ (Binomial theorem)

$(a+b)^n = b^n \left(1 + \frac{a}{b}\right)^n$

$= b^n \sum_{i=0}^{n} \binom{n}{i} a^i b^{-i}$

$= \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$ $\square$

Theorem. $\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$

(could also bijection rule + sum rule)

Proof. $\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = \sum_{i=0}^{n} \binom{n}{i} \cdot 1^i$

(by binomial theorem) $= (1+1)^n$

$= 2^n$ $\square$

Thm $\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \cdots$

Proof $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \binom{n}{4} + \cdots =$

$\sum_{i=0}^{n} \binom{n}{i} (-1)^i = (1-1)^n$ (by binomial theorem)

$= 0$ $\square$

Principle of Inclusion Exclusion (PIE)

Update on sum rule

For all A, B, $|A \cup B| = |A| + |B| - |A \cap B|$

(If A, B are disjoint, gives back sum rule)

Ex How many numbers from 1 to 100 are divisible by 2 or 5?

Ans = 60

Proof. $A = \{x \mid 1 \le x \le 100, x \text{ is divisible by } 2\}$

$B = \{x \mid 1 \le x \le 100, x \text{ is divisible by } 5\}$

$A \cap B = \{x \mid 1 \le x \le 100, x \text{ is divisible by } 2 \text{ and } 5\}$

x is divisible by 10

$|A| = 50$, $|B| = 20$, $|A \cap B| = 10$,

$|A \cup B| = |A| + |B| - |A \cap B| = 50 + 20 - 10$ (by PIE)

$= 60$

$= 0$ by 5 }   (by PIE) $= 60$

$\square$

$A \cap B = \{x \mid 1 \le x \le 100,$ x is divisible by 2 and 5$\}$

x is divisible by 10