April 24

Friday, April 24, 2020 2:10 PM

Comp Sci 212

1. Counting Rules ctd.

2. Permutations / Combinations

Announcements

- Plan for midterm/final

Generalized Product Rule

If S is a set of length-k sequences (of form $(s_1, s_2, \dots, s_k)$),

$n_1 = \#$ of choices for s_1

$n_2 = \#$ of choices for s_2 given s_1

$\vdots$

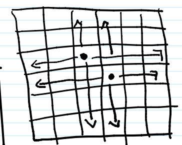
$n_k = \#$ of choices for s_k, given $s_1, \dots, s_{k-1}$

$|S| = n_1 \cdot n_2 \cdot \dots \cdot n_k$

Ex. How many ways are there to place k rooks on a $k \times k$ chessboard?

every row has at most 1 rook

every column has at most 1 rook



Use generalized product rule

$n_1 = k^2$

$n_2 = k^2 - 2k + 1 = (k-1)^2$

$n_3 = (k-2)^2$

$\vdots$

$n_i = (k-i+1)^2$

$\vdots$

$n_k = 1^2$

$\#$ ways $= n_1 \cdot n_2 \cdots n_k$

$= k^2 (k-1)^2 (k-2)^2 \cdot \dots \cdot 1^2$

$= (k \cdot (k-1) \cdot (k-2) \cdots 1)^2 = (k!)^2$

Def. Permutation - a length k sequence of elements from a set of size k, no two elements in the sequence are the same.

$S = \{a, b, c, d\}$ permutations - (c, b, a, d), (a, b, c, d)

Thm. $\#$ of permutations of a set S of size k is $k!$

Proof. Use generalized product rule. $n_1 = k$, $n_i = k - i + 1$ (if the first $i-1$ elements are $s_1, \dots, s_{i-1}$, then s_i can be anything in S besides these),

$\#$ of permutations $= n_1 n_2 n_3 \cdots n_k = k \cdot (k-1) \cdot (k-2) \cdots 1 = k!$

$\square$

Def. $k! = k \cdot (k-1) \cdot (k-2) \cdots 1$

Division Rule

Def. $f: A \to B$, is k to 1 if $|f^{-1}(b)| = k$ for every $b \in B$. $(\{a \mid f(a) = b\})$

$(k > 1)$

Division Rule: If $f: A \to B$ is k to 1, then $|A| = k \cdot |B|$

Thm. $\#$ of sequences of length ℓ from a set S of size k, no two same, is $\frac{k!}{(k-\ell)!} = {}_kP_\ell$

Proof. $A =$ set of permutations of S $\quad a \in A$, a is permutation

$B =$ set of length-ℓ sequences from S, $b \in B$ no two same.

$f: A \to B$, $f(a) =$ first ℓ elements of a.

$f^{-1}(b) = \{a \mid f(a) = b\}$

$= \{\underbrace{b \circ c}_{\text{concatenation}} \mid c \text{ is a permutation of } S \setminus b\}$ viewed as a set

$|f^{-1}(b)| = \#$ of permutations of $S \setminus b$

$= |S \setminus b|! = (k-\ell)!$

By division rule $\to$ $|B| = \frac{k!}{(k-\ell)!}$

$(|B|(k-\ell)! = |A|)$

Viewed as a set

b (viewed as a set) $= \{s \mid b_i = s \text{ for some } i\}$

Sorting permutation $\to$ sorted list

$\hookrightarrow$ a sequence of swaps

Sorting algorithm must do a diff. sequence of swaps for each permutation

f: set of permutations $\to$ order of swaps must be an injection.

$\#$ of permutations is $k!$

$B \subseteq \{0,1\}^\ell$ $\to$ running time of doing swaps $\to$

$k! \le |B| \le 2^\ell$

$\to \ell = \Omega(k \log k)$

Thm. # of subsets of size ℓ of S s.t. $|S|=k$ is $\frac{k!}{\ell!(k-\ell)!} = \binom{k}{\ell}$ (# of ways to chose ℓ elements from a set of size k)

Prof. A = set of length ℓ sequences from S, no two are same.

B = subsets of S of size ℓ

$f: A \to B$, $f(a)$ = a viewed as a set

$f^{-1}(b)$ = set of permutations of b

$|f^{-1}(b)| = \ell!$, by division rule, $|B| = \frac{k!}{\ell!(k-\ell)!}$ □

$b = (1, 1, 0, 0, \ldots 0)$

$f(b) = \{1, 2\}$

$f^{-1}(\{1,2\}) = (1, 1, 0, 0, \ldots, 0)$

Thm. # of sequences in $\{0,1\}^{10}$ w/ 2 1's is # of subsets of $[10] = \{1, 2, \ldots, 10\}$ of size $2 = \binom{10}{2}$

Prof. B = set of sequences in $\{0,1\}^{10}$ w/ 2 1's

C = set of subsets of $[10]$ of size 2.

$f: B \to C$ $\quad f(b) = \{i \mid b_i = 1\}$

if $c \in C$, $f^{-1}(c) = \{(b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9, b_{10})$

$c = \{c_1, c_2\}$ $\quad b_{c_1} = 1$, $b_{c_2} = 1$, rest are all 0

Therefore, $|f^{-1}(c)| = 1$, and f is bijection, $|B| = |C|$ □

# of ways to buy n donuts, k types = # of strings of length $n+k-1$, w/ $k-1$ 1's = $\binom{n+k-1}{k-1}$