April 20

Monday, April 20, 2020 2:37 PM

Comp Sci 212

1. Relations / Functions

2. Asymptotics

Announcements

- Survey (link on canvas)

Relation - subset of $A \times B$

Functions - relation for every $a \in A$, appears in at most one pair. Corresponding element $f(a)$, $(a, f(a)) \in$ subset.

$F = \{(a, b) \mid a \in [n], b \in [2n], b = 2a\} \subseteq [n] \times [2n]$

$[n] = \{1, 2, 3, \ldots, n\}$

$[n] \times [2n]$ = set of pairs, (a, b), $a \in [n]$, $b \in [2n]$

$f : [n] \to [2n]$, $f(a) = 2a$

$(1, 2) \in F$, $(1, 3) \notin F$

$G =$

$\{(a, b) \mid a \in [n], b \in \{\text{even}, \text{odd}\}$, a is "b" $\} \subseteq [n] \times \{\text{even}, \text{odd}\}$

$[n] \times \{\text{even}, \text{odd}\}$ = set of pairs, (a, b), $a \in [n]$, $b \in \{\text{even}, \text{odd}\}$

$\updownarrow$

$g : [n] \to \{\text{even}, \text{odd}\} = \{\text{odd}, \text{even}\}$ $\quad g(a) = \begin{cases} \text{even} & \text{if } a \text{ is even} \\ \text{odd} & \text{if } a \text{ is odd} \end{cases}$

$(1, \text{odd}) \in G$, $(1, \text{even}) \notin G$

Injections - if every $b \in B$ appears at most once

Surjection - if every $b \in B$ appears at least once — $f^{-1}(b) = \{a \mid f(a) = b\}$

$|f^{-1}(b)| \geq 1$ for all $b \in B$

→ proof, if $f(a) = f(b)$, then $a = b$

$f(a) = f(b) \to 2a = 2b \to a = b$

$f^{-1}(1) = \phi$, so f is not a surjection.

g is surjective if $n \geq 2$,

$g^{-1}(\text{even}) = \{a \mid a \in [n], a \text{ is even}\}$

$g^{-1}(\text{odd}) = \{a \mid a \in [n], a \text{ is odd}\}$

both sets are not empty if $n \geq 2$

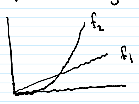
Def. Given a function $f : A \to B$ inverse image is f^{-1},

$f^{-1}(b) = \{a \mid f(a) = b\}$ (last class)

Asymptotics

Question - what is the behavior of $f(n) : \mathbb{R} \to \mathbb{R}$ as n becomes large?

$f_1(n) = 100 n \log n \qquad f_2 = 2n^2 - n$



Def. $f(n) = O(g(n))$ if $\limsup_{n \to \infty} \left| \frac{f(x)}{g(x)} \right| < \infty$ (for this class, lim)

$f(n) = o(g(n))$ if $\limsup_{n \to \infty} \frac{f(x)}{g(x)} = 0$

$f(n) = \Omega(g(n))$ if $g(n) = O(f(n))$

$f(n) = \omega(g(n))$ if $g(n) = o(f(n))$

$f(n) = \Theta(g(n))$ if $f(n) = O(g(n))$, $g(n) = O(f(n))$

Ex. $2n^2 + n = \Theta(n^2) \to \lim_{n \to \infty} \frac{2n^2 + n}{n^2} = 2$

$\lim_{n \to \infty} \frac{n^2}{2n^2 + n} = \frac{1}{2}$

$n = o(n^2)$ $\quad \lim_{n \to \infty} \frac{n}{n^2} = 0$

$1 = \omega\left(\frac{1}{\log n}\right)$ $\quad \lim_{n \to \infty} \frac{1}{\log n} = 0$

- Ignoring constant factors
- Ignoring lower order terms

Ex. $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 = \Theta(x^n)$

Fact If $a < b$, then $\lim_{n \to \infty} \frac{n^a}{n^b} = 0$.

Thm. For any $a, b > 0$ $\lim_{n \to \infty} \frac{(\log(n))^a}{n^b} = 0$

$(\log(n))^{1000000000} = o(n^{0.000001})$

Proof. Assume $a = 1$, by l'Hôpital's rule

$\lim_{n \to \infty} \frac{\log(n)}{n^b} = \lim_{n \to \infty} \frac{\frac{1}{n}}{b n^{b-1}} = 0$ if $b > 0$

If $a \neq 1$, $\lim_{n \to \infty} \frac{(\log(n))^a}{n^b} = \left(\lim_{n \to \infty} \frac{\log(n)}{n^{\frac{b}{a}}} \right)^a = 0$

$\square$

Ex. $2^n = o(4^n)$, $\lim_{n \to \infty} \frac{2^n}{4^n} = \frac{1}{2^n} = 0$.

Plain English defs

$f(n) = O(g(n))$ - $g(n)$ grows as fast or faster than $f(n)$

$f(n) = o(g(n))$ - $g(n)$ grows faster than $f(n)$

$f(n) = \Theta(g(n))$ - $g(n)$ grows as fast as $f(n)$

$f(n) = \omega(g(n))$ - $f(n)$ grows faster than $g(n)$

$f(n) = \Omega(g(n))$ - $f(n)$ grows faster than or as fast $g(n)$

If $\lim_{n \to \infty} \frac{f(x)}{g(x)} = 0$, then $f(x) = O(g(x))$, $f(x) = o(g(x))$ — $g(x)$ grows faster

If $\lim_{n \to \infty} \frac{g(x)}{f(x)} = 0$, then $f(x) = \Omega(g(x))$, $f(x) = \omega(g(x))$ - $f(x)$ grows faster

If $\lim_{n \to \infty} \frac{f(x)}{g(x)} = C$, $C \neq 0$, $f(x) = O(g(x))$, $f(x) = \Omega(g(x))$, - two grow the same

$f(x) = \Theta(g(x))$