April 15

Wednesday, April 15, 2020 2:05 PM

Comp Sci 212

1. Strong Induction
2. Invariants

Announcements

- Homework 1 is out, due April 22
  (Collaboration policy, optional problem is <u>not</u> extra credit)

Strong Induction

- Prove $P(0)$ - Base case
- Prove for all non-negative integers n, if $P(0), P(1), \ldots P(n)$, then $P(n+1)$
- Then $P(n)$ is true for all non-negative integers n

Setting - McDonalds sell chicken mcnuggets in packs of 3 and 7.

For what integers n can you buy exactly n chicken mcnuggets

($n=11$ impossible)

Thm. For all integers $n \geq 12$ you can buy exactly n chicken mcnuggets.

$n=12$ - 3, 3, 3, 3
$n=13$ - 7, 3, 3
$n=14$ - 7, 7
$n=15$ - 3, 3, 3, 3, 3
$n=16$ - 7, 3, 3, 3
$n=17$ - 7, 7, 3

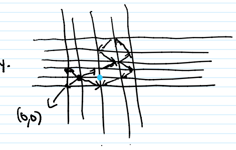
Proof. Use Induction

Base Case - $n=12$, 3, 3, 3, 3
$n=13$, 3, 3, 7
$n=14$ 7, 7

Inductive step - Assume that it's possible to buy $n-3$ chicken mcnuggets, ($n \geq 15$). Use that solution, buy extra pack of 3.

Def. Invariant is a property that stays the same over time.

Setting - Bishop on a grid, starts at (1,1). Can only move diagonally. Can the bishop get to (2,1)?



Corollary: No, because $1+1=2$, even, $2+1=3$ odd.

Thm. Let (x_t, y_t) be the position of the bishop at time t. Then $x_t + y_t$ is always even.

$(1,1) \to (0,2) \to (1,3) \to (2,4) \to (3,3)$
2 2 4 6 6

Proof. Use Induction.

Base case: $t=0$, $x_0=1$, $y_0=1$, $x_0+y_0=2$, even.

Inductive step: Assume x_t+y_t is even. Then,

Case 1: $x_{t+1}=x_t+1$, $y_{t+1}=y_t+1$
$x_{t+1}+y_{t+1}=x_t+y_t+2$, even

Case 2: $x_{t+1}=x_t+1$ $y_{t+1}=y_t-1$
$x_{t+1}+y_{t+1}=x_t+y_t$, even

Case 3: $x_{t+1}=x_t-1$, $y_{t+1}=y_t+1$
$x_{t+1}+y_{t+1}=x_t+y_t$, even

Case 4: $x_{t+1}=x_t-1$, $y_{t+1}=y_t-1$
$x_{t+1}+y_{t+1}=x_t+y_t-2$, even

$\square$

General Strategy - Find a property

$P(t)$ is true if property is satisfied at time t.

Show that $P(t)$ is true for all t by induction

In computer science loops

```
array or list A
for i 1 to A.length
    do something
end
```

Invariant - after time i, subarray from 1 to i has some property. → sorted

After loop runs - entire array has property.

(insertion, selection loop)

Puzzle

$n_0=0$

1	2	3
4	5	6
7	8	

Is it possible to get to →

$n_t=1$

1	2	3
4	5	6
8	7	

?

Invariant - # of pairs out of order will always be even

n_t = # of pairs at time t (n_t is always even)

→ (1, 2, 3, 4, 5, 6, 7, 8)

2	1	4
5	8	6
3		7

→ (2, 1, 4, 5, 8, 6, 3, 7)

(1, 2) (3, 6) (7, 8)
(3, 4) (3, 8)
(3, 5) (6, 8)

$n_t=7$
(unreachable)

Thm. n_t is always even. (sketch)

Base case - $n_0=0$

Row move - move in a row, this keeps order the same.

Column move - move within a column

↳ Case 1 -

-	-	-
-	-	a
b	c	

column move down, in rightmost column

$(a,b) \to (b,a)$
(a,c) (c,a)

exactly one is out of order

Change always be +2, 0, or -2

$(a<b, c)$ $(a>b, c)$

This proves that if n_t is even, then n_{t+1} is even

(other cases)

$\square$