April 10

Friday, April 10, 2020 2:07 PM

Comp Sci 212

1. Iff
2. Contradiction
3. Induction

Iff P iff Q → If P then Q, and if Q then P

Theorem. Let $a, b \geq 0$. Then,

$$\frac{a+b}{2} = \sqrt{ab} \text{ iff } a=b$$

Proof. If $a=b$, then $\frac{a+b}{2} = a$, $\sqrt{ab} = a$, so the two are equal.

If $\frac{a+b}{2} = \sqrt{ab}$, then $(a+b)^2 = 4ab$, therefore $a^2+2ab+b^2 = 4ab$, therefore $a^2-2ab+b^2=0$, therefore $(a-b)^2=0$, so $a-b=0$, $a=b$ □

Chain of iffs is fine

Def x is even if $\frac{x}{2}$ is an integer

Def x is odd if $\frac{x-1}{2}$ is an integer

Thm. Let x be an integer. Then x is even iff x^2 is even.

Proof. If x is even, $\frac{x}{2} = k$ for some integer k, therefore $\frac{x^2}{4} = k^2$, $\frac{x^2}{2} = 2k^2$, $2k^2$ is an integer, so x^2 is even.

If x is odd, $\frac{x-1}{2} = k$ for some integer k, therefore $\frac{(x-1)^2}{4} = k^2 \rightarrow \frac{x^2-1}{2} = 2k^2+x-1$, therefore x^2 is odd.

Contradiction P is true → If not P then Q

If not P then not Q

If not P then P

Theorem. $\sqrt{2}$ is irrational

Proof. Use contradiction. Assume $\sqrt{2}$ is rational. → If not P then Q

$\sqrt{2} = \frac{m}{n}$, m, n are integers, lowest terms, not the case that both are even.

$2 = \frac{m^2}{n^2} \rightarrow \underline{2n^2 = m^2} \rightarrow m^2$ is even $\rightarrow m$ is even ← If not P then not Q

$\rightarrow m^2$ is divisible by 4 $\rightarrow n^2$ is even $\rightarrow n$ is even, ($\frac{m^2}{2}$ is even)

contradicts the fact that m and n are not both even. □

$\frac{m^2}{4} = k$ – integer

$\frac{\left(\frac{m^2}{2}\right)}{2} = k$, $\left(\frac{m^2}{2}\right) = n$

Theorem. There exist an infinite number of primes.

numbers only divisible by 1 & itself

2, 3, 5, 7, 11, 13

not 9 not 15

Proof. Assume there exist a finite # of primes n primes

$p_1, p_2, p_3, \ldots, p_n$

$p_1 p_2 p_3 \cdots p_n + 1$, not divisible by $p_1, p_2, \ldots, p_n$ (multiplication) (assume all numbers can be written as a product of primes) must also be prime. But this contradicts the fact that there are only n primes □

Induction

Theorem For all non-negative integers n,

$$1+2+3+\cdots+n = \frac{n(n+1)}{2}$$

$n=1$ 1 $\frac{1 \cdot 2}{2} = 1$

$n=2$ $1+2=3$ $\frac{2 \cdot 3}{2} = 3$

$n=3$ $1+2+3=6$ $\frac{3 \cdot 4}{2} = 6$

$n=4$ $1+2+3+4=10$ $\frac{4 \cdot 5}{2} = 10$

$n=5$ $\underline{1+2+3+4}+5$

$10+5=15$

$$\frac{4 \cdot 5}{2} + 5 = \frac{4 \cdot 5 + 2 \cdot 5}{2} = \frac{6 \cdot 5}{2}$$

Idea: Use solution for smaller n to prove solution for larger n.

Strategy: Predicate, P(n) = statement is true for n

Prove P(0) – Base case

Prove for all n, if P(n) then P(n+1) – Inductive step

Then P(n) is true for all non-negative integers n (prove on Monday)

Use Induction

Proof. Base case: $n=0$ $0 = \frac{0 \cdot 1}{2}$ ✓

($n=1$ $1 = \frac{1 \cdot 2}{2}$ ✓ also works)

Inductive step: Assume $1+\cdots+n = \frac{n(n+1)}{2}$.

Then, $\underbrace{1+\cdots+n}+n+1 = \frac{n(n+1)}{2} + n+1$

$$= \frac{n(n+1)+2(n+1)}{2}$$

$$= \frac{(n+1)(n+2)}{2}$$ ✓

□