February 26
Wednesday, February 26, 2020 3:48 PM

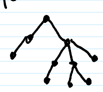
Comp Sci 212

1. Trees

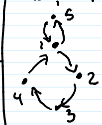
Def. neighbor, u is a neighbor of v in a graph $G=(V,E)$ if $\{u,v\} \in E$

Theorem. The following are equivalent ($G=(V,E)$ a graph)

1. G is connected and acyclic
2. Every pair of vertices is connected by a unique path (repeats not okay)
3. G is connected and $|E|=|V|-1$ (def. of tree)
4. G is acyclic and $|E|=|V|-1$
5. G is acyclic and adding any new edge creates a cycle.



Iff, 1 + 2 are equivalent $1 \to 2$, $2 \to 1$



Proof. $4 \to 1$

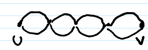
Monday - If a graph is acyclic, it has exactly $|V|-|E|$ connected components.

If G is acyclic, $|E|=|V|-1$, then G has 1 connected component, therefore G is connected + acyclic.

$1 \to 2$ Use contradiction, assume there exist u, v s.t. u + v have more than 1 path connecting them. - Because G is connected, there must be at least 1 path between u + v.

$(u, x_1, x_2, \ldots, x_n, v)$ - path 1

$(u, y_1, y_2, \ldots, y_m, v)$ - path 2



Let i be the largest # s.t. x_i is in path 1 + path 2. Then $(v, x_n, x_{n-1}, \ldots, x_i, y_{j+1}, y_{j+2}, \ldots, y_m, v)$ is a cycle. ($= y_j$)

$2 \to 3$ G is connected, by definition.

Need to show $|E|=|V|-1$.

Use strong induction on the # of vertices

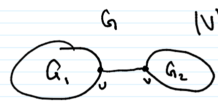
Base Case. $|V|=1$, no edges

Inductive step: Assume the statement is true for trees w/ less than $|V|$ vertices.

Pick an arbitrary $u \in V$, let v be a neighbor of u

$G' = (V, E \setminus \{u,v\})$, u + v must be disconnected

Thus, G' has 2 connected components, G_1, G_2 $G_i = (V_i, E_i)$



G_1, G_2 both satisfy 2 (by induction hypothesis)

$|E_1| = |V_1|-1$, $|E_2| = |V_2|-1$

$|V| = |V_1| + |V_2|$,

$|E| = |E_1| + |E_2| + 1 =$

$|V_1| - 1 + |V_2| - 1 + 1 = |V| - 1$

□

$5 \to 1$ Use contradiction, assume u + v in G are not connected,

Adding edge $\{u,v\}$ must create a cycle, let the cycle be $(u, x_1, x_2, \ldots, v, u)$, (cycle must contain $\{u,v\}$, because G is acyclic), this contradicts assumption that u + v are not connected.

$1 \to 5$ Let $\{u,v\}$ be a new edge,

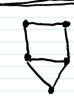
Let $(u, x_1, x_2, \ldots, v)$ be a path from u to v in G, must exist because G is connected. Then $(u, x_1, x_2, \ldots, v, u)$ is a cycle, a new cycle because G is acyclic.

$3 \to 4$ Use contradiction, assume that G (sketch) contains a cycle, k edges, k vertices.

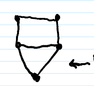
Start with a cycle. Add vertices one by one, w/ corresponding edges to vertices already there.

We can do this because G is connected.

Each time a vertex is added, you add at least 1 edge. This implies that $|E| \geq |V|$, a contradiction.



← graph w/ cycles



← process

step	# edges	# vertices
1	3	3
2	4	4
3	6	5