February 19

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Comp Sci 212

1. Chebyshev's Inequality

2. Chernoff Bounds

Markov's Inequality

Thm. If R is a non-negative r.v., $x>0$ then

$$\Pr[R \geq x] \leq \frac{\mathbb{E}[R]}{x}$$

$\mathbb{E}[R]=50$

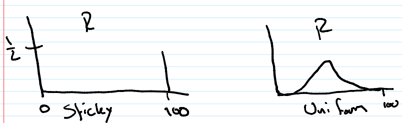
$S' = \{H, T\}$, $S = \underbrace{S' \times S' \times \cdots \times S'}_{100 \text{ times}}$

uniform

sticky – all heads / all tails

$R: S \to \mathbb{R}$, $R(s) = \#$ of heads in s

Markov – $\Pr[R \geq 75] \leq \frac{50}{75} = \frac{2}{3}$, both cases



Instead of looking at $\mathbb{E}[R]$, try $\mathbb{E}[R^2]$

$R^2(s) = (R(s))^2$

Sticky – $\mathbb{E}[R^2] = 5{,}000$

Uniform – $E_i = i^{th}$ coin is heads

$R = \sum_{i=1}^{100} \mathbf{1}_{E_i}$ $\qquad R^2 = \left(\sum_{i=1}^{100} \mathbf{1}_{E_i}\right)^2$

$$\mathbb{E}[R^2] = \mathbb{E}\left[\left(\sum_{i=1}^{100} \mathbf{1}_{E_i}\right)^2\right]$$

$$= \mathbb{E}\left[\sum_{i,j=1}^{100} \mathbf{1}_{E_i}\mathbf{1}_{E_j}\right] = \sum_{i,j=1}^{100} \mathbb{E}[\mathbf{1}_{E_i}\mathbf{1}_{E_j}]$$

$\frac{1}{4}$ if $i \neq j$, $\frac{1}{2}$ if $i=j$

$$= \sum_{i=1}^{100} \mathbb{E}[\mathbf{1}_{E_i}\mathbf{1}_{E_i}] + \sum_{\substack{i,j \text{ s.t.} \\ i \neq j}} \mathbb{E}[\mathbf{1}_{E_i}\mathbf{1}_{E_j}]$$

$$= \sum_{i=1}^{100} \frac{1}{2} + \sum_{\substack{i,j \text{ s.t.} \\ i \neq j}} \frac{1}{4} = \frac{100}{2} + \frac{9900}{4} = 2525$$

Chebyshev's Inequality

Thm. R is a r.v., x is positive

$$\Pr[|R - \mathbb{E}[R]| \geq x] \leq \frac{\mathbb{E}[R^2] - \mathbb{E}[R]^2}{x^2} \left(\frac{\mathrm{Var}[R]}{x^2}\right)$$

Proof. $Y = (R - \mathbb{E}[R])^2$

$$\Pr[|R - \mathbb{E}[R]| \geq x] = \Pr[Y \geq x^2] \leq \frac{\mathbb{E}[Y]}{x^2}$$

Need to show $\mathbb{E}[Y] = \mathbb{E}[R^2] - \mathbb{E}[R]^2$

$$\mathbb{E}[Y] = \mathbb{E}[R^2 - 2R\mathbb{E}[R] + \mathbb{E}[R]^2]$$
$$= \mathbb{E}[R^2] - \mathbb{E}[2R\mathbb{E}[R]] + \mathbb{E}[\mathbb{E}[R]^2]$$
$$= \mathbb{E}[R^2] - 2\mathbb{E}[R]\mathbb{E}[R] + \mathbb{E}[R]^2$$
$$= \mathbb{E}[R^2] - \mathbb{E}[R]^2$$

□

$$\Pr[R \geq 75] \leq \Pr[|R - \mathbb{E}[R]| \geq 25]$$

Uniform $\leq \frac{2525 - 2500}{25^2} = \frac{1}{25}$

Sticky $\leq \frac{5000 - 2500}{25^2} = 4$

Def. Variance,

$$\mathrm{Var}[R] = \mathbb{E}[R^2] - \mathbb{E}[R]^2$$

(Informal) – a measure of how spread out a distribution is

Thm. If R_1 and R_2 are independent random variables, $\mathrm{Var}[R_1 + R_2] = \mathrm{Var}[R_1] + \mathrm{Var}[R_2]$

Proof. $\mathrm{Var}[R_1 + R_2] = \mathbb{E}[(R_1+R_2)^2] - (\mathbb{E}[R_1+R_2])^2$ (by def.)

$= \mathbb{E}[R_1^2 + 2R_1R_2 + R_2^2] - (\mathbb{E}[R_1] + \mathbb{E}[R_2])^2$

$= \mathbb{E}[R_1^2] + 2\mathbb{E}[R_1R_2] + \mathbb{E}[R_2^2] - \mathbb{E}[R_1]^2 - \mathbb{E}[R_2]^2 - 2\mathbb{E}[R_1]\mathbb{E}[R_2]$

Cancel because $\mathbb{E}[R_1R_2] = \mathbb{E}[R_1]\mathbb{E}[R_2]$ ind.

$= \mathbb{E}[R_1^2] - \mathbb{E}[R_1]^2 + \mathbb{E}[R_2^2] - \mathbb{E}[R_2]^2$

$= \mathrm{Var}[R_1] + \mathrm{Var}[R_2]$

□

Moment bounds

Thm. If q is even, $\mathbb{E}[R] = 0$, then x is positive

$$\Pr[|R| \geq x] \leq \frac{\mathbb{E}[R^q]}{x^q} \longleftarrow \text{moment}$$

Chernoff bounds $\quad t, x$ are positive

$$\Pr[R \geq x] \leq \frac{\mathbb{E}[e^{tR}]}{e^{tx}}$$

Proof. Moment $Y = R^q$

$\Pr[|R| \geq x] = \Pr[R^q \geq x^q]$, use Markov

Chernoff bounds, $Y = e^{tR}$

$\Pr[R \geq x] = \Pr[e^{tR} \geq e^{tx}]$, use Markov

□

Chernoff

Thm. If $R_1, R_2, \ldots, R_n$ are independent random variables s.t. $R_i: S \to [-1, 1]$

$\mathbb{E}[R_i] = 0$, for all i, u positive

$$\Pr[R_1 + \cdots + R_n \geq u\sqrt{n}] \leq e^{\frac{-u^2}{4e}}$$