February 7

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Comp Sci 212

1. Pairwise Independence
2. Birthday Paradox
3. Random Variables

Announcements

- Midterm February 10, LR2

Def. Pairswise independence - $E_1, \dots, E_n$ are pairwise independent if for every $i, j \in [n]$

$Pr[E_i \cap E_j] = Pr[E_i] Pr[E_j]$

Ex. Flip 2 coins - uniform distribution

$S' = \{Heads, Tails\}$, $S' \times S' = \{(H,H), (H,T), (T,H), (T,T)\}$

E_1 = first coin heads $= \{(H,H), (H,T)\}$

E_2 = second coin heads $= \{(H,H), (T,H)\}$

E_3 = both coins same $= \{(H,H), (T,T)\}$

$Pr[E_1] = \frac{|E_1|}{|S|} = \frac{1}{2}$ $\quad E_1 \cap E_2 = E_1 \cap E_3 = E_2 \cap E_3 = \{(H,H)\}$

$Pr[E_2] = \frac{|E_2|}{|S|} = \frac{1}{2}$ $\quad Pr[E_1 \cap E_2] = \frac{1}{4} = Pr[E_1] \cdot Pr[E_2]$

$Pr[E_3] = \frac{|E_3|}{|S|} = \frac{1}{2}$ $\quad Pr[E_1 \cap E_3] = \frac{1}{4} = Pr[E_1] \cdot Pr[E_3]$

$Pr[E_2 \cap E_3] = \frac{1}{4} = Pr[E_2] Pr[E_3]$

- Pairwise independent

$Pr[E_1 \cap E_2 \cap E_3] = \frac{|E_1 \cap E_2 \cap E_3|}{|S|} = \frac{|\{(H,H)\}|}{4} = \frac{1}{4}$

$\neq Pr[E_1] Pr[E_2] Pr[E_3] = \frac{1}{8}$

Not mutually independent

Birthday Paradox

In a room of n people, whats the probability that two people have the same birthday?

- D = set of possible birthdays
- Distribution of birthdays is uniform over $\underbrace{D \times D \times \cdots \times D}_{n \text{ times}} = D^n$
- E = set of sequences no two are the same.

$Pr[E] = \frac{|E|}{|D^n|} = \frac{|D|}{|D|} \frac{(|D|-1)}{|D|} \frac{(|D|-2)}{|D|} \cdots \frac{(|D|-n+1)}{|D|}$

$= 1 \cdot \left(1 - \frac{1}{|D|}\right)\left(1 - \frac{2}{|D|}\right) \cdots \left(1 - \frac{n-1}{|D|}\right)$ $\quad 1 - x \le e^{-x}$

$\le e^0 \cdot e^{-\frac{1}{|D|}} e^{-\frac{2}{|D|}} \cdots e^{-\frac{n-1}{|D|}} = e^{-\frac{1}{|D|}(0+1+2+\dots+(n-1))}$

$= e^{-\frac{n(n-1)}{2|D|}} \longrightarrow$ If $\frac{n(n-1)}{2} \ge |D|$, or $n \ge \sqrt{|D|}$, at most $\frac{1}{e}$

Birthday Principle

If there are d days in a year, and $\sqrt{2d}$ people in a room, then the probability that at least 2 share a birthday is at least $1 - \frac{1}{e}$

Def. A random variable is a total function on a probability space.

Ex. $S' = \{Heads, Tails\}$, $S = \underbrace{S' \times S' \times \dots \times S'}_{100 \text{ times}}$

$R: S \to \mathbb{R}$, $R(s)$ = # of times heads appears in s.

$E = \{s \mid s \in S \text{ s contains 50 heads}\}$

$Pr[R=50] = Pr[E]$

uniform - $\frac{|E|}{|S|} = \frac{\binom{100}{50}}{2^{100}} = Pr[R=50]$

sticky (all heads or all tails) - $0 = Pr[R=50]$

$M: S \to \mathbb{R}$,

$M(s) = \begin{cases} 1 & \text{if all coins are the same} \\ 0 & \text{o.w.} \end{cases}$

$Pr[M=1]$, $E = \{\text{all heads, all tails}\}$

$Pr[M=1] = Pr[E]$

uniform - $Pr[M=1] = \frac{|E|}{|S|} = \frac{2}{2^{100}} = 2^{-99}$

sticky - $Pr[M=1] = 1$