January 29

Wednesday, January 29, 2020 3:02 PM

Comp Sci 212

1. Binomial Theorem
2. Pigeonhole Principle
3. Principle of Inclusion - Exclusion

Announcements

- Midterm February 10 LR2

Thm. $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$

$\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Binomial Theorem

Thm. For $n \in \mathbb{N}$, $(1+x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i$

$= \binom{n}{0}x^0 + \binom{n}{1}x^1 + \dots + \binom{n}{n}x^n$

Proof. Use Induction

Base case $n=0$: $1 = \binom{0}{0}x^0$, where $\binom{0}{0} = 1$

Inductive step - Assume $(1+x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i$

Then $(1+x)^{n+1} = (1+x)(1+x)^n$

$= (1+x)\left(\sum_{i=0}^{n} \binom{n}{i} x^i\right)$ (By inductive hypothesis)

(Compute the coefficient of x^i by the bijection rule)

$= \sum_{i=0}^{n} \binom{n}{i} x^i + \sum_{i=0}^{n} \binom{n}{i} x^{i+1}$

$= \sum_{i=0}^{n} \binom{n}{i} x^i + \sum_{i=1}^{n+1} \binom{n}{i-1} x^i$

$\binom{n}{k} = 0$ if $k<0$, or $k>n$ $\qquad + \binom{n}{n+1}x^{n+1} + \binom{n}{-1}x^{-1}$

$= \sum_{i=0}^{n+1} \binom{n}{i} x^i + \sum_{i=0}^{n+1} \binom{n}{i-1} x^i$

$= \sum_{i=0}^{n+1} x^i \left(\binom{n}{i} + \binom{n}{i-1}\right)$

$= \sum_{i=0}^{n+1} x^i \binom{n+1}{i}$

□

Corollary. If $n \in \mathbb{N}$, $(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$

Proof. $\left(1+\frac{a}{b}\right)^n = \sum_{i=0}^{n} \binom{n}{i} \frac{a^i}{b^i}$ (by binomial theorem)

$(a+b)^n = b^n\left(1+\frac{a}{b}\right)^n$

$= b^n \sum_{i=0}^{n} \binom{n}{i} \frac{a^i}{b^i}$

$= \sum_{i=0}^{n} \binom{n}{i} a^i \cdot b^{n-i}$

□

Multinomial Theorem

Theorem $(z_1 + z_2 + \dots z_m)^n = \sum_{\substack{k_1, k_2, \dots k_m \in \mathbb{N} \\ k_1+k_2+\dots+k_m = n}} \frac{n!}{k_1! k_2! \dots k_m!} z_1^{k_1} z_2^{k_2} \dots z_m^{k_m}$

Sum over all elements of the set $\{(k_1, \dots, k_m) \mid k_i \in \mathbb{N}, k_1 + \dots k_m = n\}$

If $m=2$, $k_2 = n-k_1$ get $\sum_{k_1 \in \mathbb{N}} \frac{n!}{k_1!(n-k_1)!} z_1^{k_1} z_2^{n-k_2}$

$= \sum_{k_1 \in \mathbb{N}} \binom{n}{k_1} z_1^{k_1} z_2^{n-k_2}$

Theorem. $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots \binom{n}{n} = 2^n$

Proof. $2^n = (1+1)^n = \sum_{i=0}^{n} \binom{n}{i} 1^i 1^{n-i} = \sum_{i=0}^{n} \binom{n}{i}$ (by binomial theorem)

□

Theorem. $\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \dots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \dots$

Proof. $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \binom{n}{4} - \binom{n}{5} + \dots = 0$

Need to prove ↑

$0 = (-1+1)^n = \sum_{i=0}^{n} \binom{n}{i} \underbrace{1^{n-i} \cdot (-1)^i}_{\substack{1 \text{ if } i \text{ is even} \\ -1 \text{ if } i \text{ is odd}}} = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \binom{n}{4}$

(by binomial theorem)

Pigeonhole Principle

If number of pigeons in holes is greater than the number of holes, there must be at least 1 hole w/ more than one pigeon.

If $f: A \to B$, total function, $k|B| < |A|$, then exist $b \in B$ s.t. $|f^{-1}(b)| \geq k+1$

Theorem. In a sock drawer w/ blue + black socks, picking out three socks gets you a pair w/ the same color.

Proof. $S = \{s_1, s_2, s_3\}$ be the socks

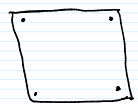
$C = \{\text{blue}, \text{black}\}$ be the possible colors.

$f: S \to C$, $f(s)$ = color of sock s

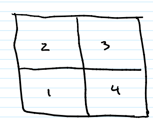
$|C| < |S|$, by the pigeonhole principle, there exists $c \in C$ such that $|f^{-1}(c)| \geq 2$

i.e., there are two socks w/ same color □

Theorem. 5 points in a 1×1 square, there exist at least 2, that are a distance of at most $\frac{\sqrt{2}}{2}$ from each other.



Proof.



Holes - $\{1, 2, 3, 4\}$

Points $\{p_1, p_2, p_3, p_4, p_5\}$

f: Points → Holes, by the pigeonhole principle at least 2 points in 1 hole → two points a distance of $\leq \frac{\sqrt{2}}{2}$