February 14

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Comp Sci 212

1. Linearity of Expectation
2. Binomial Distribution
3. Expected Value of Products

Def. (Expected Value) X is random variable, S = sample, $\Pr$ = Probability dist. $X: S \to A$ (usually A is $\mathbb{R}$)

$$\mathbb{E}[X] = \sum_{\omega \in S} X(\omega)\Pr[\omega]$$

(like mean or average)

Linearity of Expection

Thm. $\mathbb{E}[R_1 + R_2] = \mathbb{E}[R_1] + \mathbb{E}[R_2]$, R_1, R_2 are random variables

$R_1 + R_2 : S \to A$, $R_1 + R_2(\omega) = R_1(\omega) + R_2(\omega)$

Proof. $\mathbb{E}[R_1 + R_2] = \sum_{\omega \in S} R_1 + R_2(\omega)\Pr[\omega]$

$= \sum_{\omega \in S} (R_1(\omega) + R_2(\omega))\Pr[\omega]$

$= \sum_{\omega \in S} R_1(\omega)\Pr[\omega] + \sum_{\omega \in S} R_2(\omega)\Pr[\omega]$

$= \mathbb{E}[R_1] + \mathbb{E}[R_2]$ $\square$

- R_1 and R_2, don't have to be independent

Ex. 6-sided die $S = \{1, 2, 3, 4, 5, 6\}$ uniform dist.

$X: S \to \mathbb{R}$, $X(s) = s$ $\quad \mathbb{E}[X] = \frac{7}{2}$

$Y: S \to \mathbb{R}$ $\quad Y(s) = \frac{1}{s}$ $\quad \mathbb{E}[Y] = \frac{49}{120}$

$\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y] = \frac{7}{2} + \frac{49}{120}$

$X+Y(s) = s + \frac{1}{s}$

Def. Indicator random variable - Random variable whose range is $\{0,1\}$ (Usually associated w/ an event)

$\mathbf{1}_E : S \to \{0,1\}$ $\quad \mathbf{1}_E(s) = \begin{cases} 1 & \text{if } s \in E \\ 0 & \text{o.w.} \end{cases}$

Fact. $\mathbb{E}[\mathbf{1}_E] = \sum_{\omega \in S} \mathbf{1}_E(\omega)\Pr[\omega]$

$= \sum_{\omega \in E} \Pr[\omega] + \sum_{\omega \in S \setminus E} 0 = \sum_{\omega \in E} \Pr[\omega] = \Pr[E]$

$S' = \{H, T\}$, $S = \underbrace{S' \times S' \times \cdots \times S'}_{100 \text{ times}}$

two dists - uniform
sticky coin - all heads
all tails

$R: S \to \mathbb{R}$, $R(s)$ = # of heads in s

$\mathbb{E}[R] = \sum_{s \in S} R(s) \cdot \Pr[s]$ $\longrightarrow \mathbb{E}[R] = 50$ Sticky

$= \sum_{s \in S} \frac{R(s)}{2^n}$ $\leftarrow$ Uniform

E_i = i^{th} coin is heads, $= \{s \mid s \in S, s_i = H\}$

$\mathbf{1}_{E_i}(s) = \begin{cases} 1 & \text{if } i^{th} \text{ coin is heads} \\ 0 & \text{o.w.} \end{cases}$

$R = \mathbf{1}_{E_1} + \mathbf{1}_{E_2} + \cdots + \mathbf{1}_{E_{100}}$

$\mathbb{E}[R] = \mathbb{E}[\mathbf{1}_{E_1} + \mathbf{1}_{E_2} + \cdots + \mathbf{1}_{E_{100}}] = \mathbb{E}[\mathbf{1}_{E_1}] + \cdots + \mathbb{E}[\mathbf{1}_{E_{100}}]$

$= \Pr[E_1] + \Pr[E_2] + \cdots + \Pr[E_{100}]$

$= \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2} = \frac{100}{2} = 50$

Generally, $E_1, \dots, E_n$ events $R: S \to \mathbb{R}$ s.t. $R(s)$ = # of events satisfied by s

$$\mathbb{E}[R] = \sum_{i=1}^{n} \Pr[E_i]$$

Binomial distribution

$S = \{0,1\}^n$, $\Pr[s] = p^{\# \text{ of 1's in } s} \cdot (1-p)^{\# \text{ of 0's in } s}$

If $p = \frac{1}{2}$ $\Pr[s] = (\frac{1}{2})^n$, uniform dist.

$= B(n,p)$, $p \in [0,1]$

$\sum_{s \in \{0,1\}^n} p^{\# \text{ of 1's in } s} (1-p)^{\# \text{ of 0's in } s} = (p + (1-p))^n$

$= 1^n = 1$

$R: S \to \mathbb{R}$ $\quad R(s)$ = # of 1's in s

Goal: Compute $\mathbb{E}[R]$

include $\binom{n}{i}$ outcomes $\Pr[s] = p^i(1-p)^{n-i}$

$\mathbb{E}[R] = \sum_{s \in S} \Pr[s] \cdot R(s) = \sum_{i=0}^{n} i \cdot \Pr[R = i]$

$= \sum_{i=0}^{n} i \binom{n}{i} \cdot p^i \cdot (1-p)^{n-i}$

E_i = event that i^{th} character is 1

$= \{s \in \{0,1\}^n \mid s_i = 1\}$

$R = \mathbf{1}_{E_1} + \mathbf{1}_{E_2} + \mathbf{1}_{E_3} + \cdots + \mathbf{1}_{E_n}$

$\mathbb{E}[R] = \sum_{i=1}^{n} \Pr[E_i] = pn$

$\Pr[E_1] = \sum_{\substack{s \in \{0,1\}^n \\ s_1 = 1}} p^{\# \text{ of 1's in } s} (1-p)^{\# \text{ of 0's in } s}$

$= p \sum_{s \in \{0,1\}^{n-1}} p^{\# \text{ 1's in } s} (1-p)^{\# \text{ of 0's in } s}$

↳ sum of probs in $B(n-1, p)$

$= p$

Calculate $\sum_{i=0}^{n} i \binom{n}{i} p^i (1-p)^{n-i} = pn$

Fact. $\mathbb{E}[X] = \sum_{r \in \text{Range}(X)} r \cdot \Pr[X = r]$

Prof. $\mathbb{E}[X] = \sum_{\omega \in S} X(\omega)\Pr[\omega]$

$= \sum_{r \in \text{Range}(X)} \sum_{\substack{\omega \in S \\ \text{s.t.} \\ X(\omega) = r}} \Pr[\omega] \cdot X(\omega)$ — always r

$= \sum_{r \in \text{Range}(X)} r \sum_{\substack{\omega \in S \\ \text{s.t.} \\ X(\omega) = r}} \Pr[\omega] = \Pr[X = r]$

$= \sum_{r \in \text{range}(X)} r \cdot \Pr[X = r]$

Expected Value of Product

Thm. If R_1 & R_2 are <u>independent</u> random variables

$\mathbb{E}[R_1]\mathbb{E}[R_2] = \mathbb{E}[R_1 \cdot R_2]$ $\quad (R_1 \cdot R_2(\omega) = R_1(\omega) \cdot R_2(\omega))$

(If $R_1 = R_2$, R_1 die example $\mathbb{E}[R_1^2] \neq \mathbb{E}[R_1]^2$)

Prof. $\mathbb{E}[R_1 \cdot R_2] = \sum_{r \in \text{Range}(R_1 \cdot R_2)} \Pr[R_1 \cdot R_2 = r] \cdot r$

$= \sum_{\substack{r_1 \in \text{Range}(R_1) \\ r_2 \in \text{Range}(R_2)}} \Pr[R_1 = r_1 \text{ and } R_2 = r_2] \cdot r_1 \cdot r_2$

$= \sum_{r_1 \in \text{Range}(R_1)} \Pr[R_1 = r_1] \cdot r_1 \Pr[R_2 = r_2] \cdot r_2$ (by independence)

$= \left(\sum_{r \in \text{Range}(R_1)} \Pr[R_1 = r_1] \cdot r_1\right)\left(\sum_{r \in \text{Range} R_2} \Pr[R_2 = r_2] \cdot r_2\right)$ (factoring)

$= \mathbb{E}[R_1] \cdot \mathbb{E}[R_2]$ (by def.)