February 12

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Comp Sci 212

1. Random Variables
2. Expected Value
3. Mean time until failure

Announcements

- Midterm grades out
- Homework 4 out

Def. A random variable R is a total function on a probability space.

Ex. $S' = \{H, T\}$, $S = \underbrace{S' \times S' \times \cdots \times S'}_{100 \text{ times}}$

$R: S \to \mathbb{R}$ $R(s) = \{\#$ of heads in s

$Pr[R = 50] = Pr[E]$

$E = \{s \mid s \in S,\ s \text{ contains 50 heads}\}$

$Pr[R \leq 25]$, $Pr[R \text{ is even}]$ $Pr[|R-50| \leq 10]$

uniform dist

sticky dist $<$ all heads w.p. $\frac{1}{2}$ / all tails w.p. $\frac{1}{2}$

$Pr[R=50] = \frac{|E|}{|S|}$ uniform $\qquad$ 0 sticky

$= \frac{\binom{100}{50}}{2^{100}}$

Ex. $M: S \to \mathbb{R}$ $\quad M(s) = \begin{cases} 1 & \text{if all coins are the same} \\ 0 & \text{o.w.} \end{cases}$

$Pr[M=1]$, uniform dist. $= \frac{|E|}{|S|}$ – $E = \{$ all heads, all tails

$= 2^{-99}$

sticky $= 1$

Def. Two random variables A & B are independent if

$$Pr[A = x_1] \cdot Pr[B = x_2] = Pr[A = x_1 \text{ and } B = x_2]$$

for all x_1, x_2

Thm. R and M are not independent (uniform dist)

Proof. $Pr[R=2 \text{ and } M=1] = 0$, events are disjoint

$Pr[M=1] \neq 0$, $Pr[R=2] = \frac{\binom{100}{2}}{2^{100}} \neq 0$

$Pr[R=2] \cdot Pr[M=1] \neq Pr[R=2 \text{ and } M=1]$

Thm. R & M are independent (sticky dist)

Proof. $M=0$, $Pr[M=0]=0$, $Pr[R=x_1] \cdot Pr[M=0]$

$Pr[R=x_1 \text{ and } M=0] = 0 = 0$ for all x_1

$Pr[M=1]=1$, $Pr[M=1] \cdot Pr[R=x_1] =$

$Pr[R=x_1] = Pr[R=x_1 \text{ and } M=1]$

Thm. If A and B are independent,

$$Pr[A \leq x_1] \cdot Pr[B \leq x_2] = Pr[A \leq x_1 \text{ and } B \leq x_2]$$

Expected Value

Def. (Expected Value) X is a random variable, Sample space S, Prob. dist. $Pr: S \to \mathbb{R}$

$X: S \to A$ (most of the time, $\mathbb{R}$)

$\mathbb{E}[X] = \sum_{w \in S} X(s) \cdot Pr[w]$

sum over all w in S

(like mean, average)

Ex. 6-sided die $S = \{1, 2, 3, 4, 5, 6\}$, uniform dist

$X: S \to \mathbb{R}$ $\quad X(s) = s$

$\mathbb{E}[X] = \sum_{w \in S} Pr[w] \cdot X(w) = \frac{1}{6} + \frac{2}{6} + \frac{3}{6} + \frac{4}{6} + \frac{5}{6} + 1$

$= \frac{7}{2}$

$Y: S \to \mathbb{R}$ $\quad Y(s) = \frac{1}{s}$ (also $\frac{1}{X}$ – a random variable)

$\mathbb{E}[Y] = \sum_{w \in S} Pr[w] Y(w) = \frac{1}{6} \cdot 1 + \frac{1}{6} \cdot \frac{1}{2} + \frac{1}{6} \cdot \frac{1}{3} + \frac{1}{6} \cdot \frac{1}{4} + \frac{1}{6} \cdot \frac{1}{5} + \frac{1}{6} \cdot \frac{1}{6}$

$= \frac{49}{120} \neq \frac{2}{7}$

not equal to $\frac{1}{\mathbb{E}[X]}$

$Z: S \to \mathbb{R}$ $\quad Z(s) = s^2$, (also, X^2)

$\mathbb{E}[Z] = \sum_{w \in S} Pr[w] Z(w) = \frac{1}{6} + \frac{4}{6} + \frac{9}{6} + \frac{16}{6} + \frac{25}{6} + \frac{36}{6} = \frac{91}{6}$

$\neq \mathbb{E}[X]^2$

Def. (Conditional expectation) X r.v., A event

$\mathbb{E}[X \mid A] = \sum_{w \in A} X(w) Pr[w \mid A]$

$= \frac{1}{Pr[A]} \sum_{w \in A} X(w) Pr[w]$

Law of total expectation – R – random variable

$S_1, S_2, \ldots, S_n$ – disjoint ($S_i \cap S_j = \emptyset$)

$S_1 \cup S_2 \cup \cdots \cup S_n = S$ – sample space

$\mathbb{E}[R] = \sum_{i=1}^{n} \mathbb{E}[R \mid S_i] \cdot Pr[S_i]$

Proof. Like law of total probability $\square$

Mean time until failure

Each day your computer fails w.p. p, how long will it last on average?

$S = \{1, 2, 3, \ldots, \}$

Outcome i, the case where your computer lasts until day i, but fails on day i.

$Pr[i] = (1-p)^{i-1} \cdot p$, $\quad \sum_{i=1}^{\infty} p \cdot (1-p)^{i-1}$ should equal 1

$= p \sum_{i=1}^{\infty} (1-p)^{i-1}$

$= p \cdot \frac{1}{1-(1-p)} = p \cdot \frac{1}{p} = 1$

$R: S \to \mathbb{R}$, $R(w) = w$

Want to calculate is $E[R] = \sum_{i=1}^{\infty} \underbrace{p(1-p)^{i-1}}_{Pr[w]} \cdot \underbrace{i}_{R(w)}$

Let $A = \{1\}$, $Pr[A] = p$

$\mathbb{E}[R] = \mathbb{E}[R \mid A] Pr[A] + \mathbb{E}[R \mid S \setminus A] \cdot Pr[S \setminus A]$

(by law of total probability)

$= 1 \cdot p + (1 + \mathbb{E}[R])(1-p)$

$= p + (1-p) + \mathbb{E}[R](1-p)$

$= 1 + \mathbb{E}[R](1-p)$

$\downarrow$

$\mathbb{E}[R] \cdot p = 1 \to \mathbb{E}[R] = \frac{1}{p}$

$\mathbb{E}[R \mid S \setminus A] = \left(\sum_{w=2}^{\infty} R(w) \cdot Pr[w] \right) \frac{1}{Pr[S \setminus A]}$

$= \sum_{w=2}^{\infty} \left(w \cdot (1-p)^{w-1} \cdot p \right) \frac{1}{1-p}$

$= \sum_{w=2}^{\infty} w(1-p)^{w-2} \cdot p = \sum_{w=1}^{\infty} (1+w)(1-p)^{w-1} \cdot p$

$= 1 + \mathbb{E}[R]$