February 5

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Comp Sci 212

1. Conditional Probability
2. Independence
3. Birthday Paradox

Announcements

- Midterm February 10, LR2

Def. $Pr[A \mid B] = \frac{Pr[A \cap B]}{Pr[B]}$

Law of total Probability

If $E_1, E_2, \ldots, E_n$ are disjoint,

$$\sum_{i=1}^{n} Pr[A \mid E_i] Pr[E_i] = Pr[A \cap (E_1 \cup E_2 \cup \ldots \cup E_n)]$$

Useful if equal to sample space

Proof. $\sum_{i=1}^{n} Pr[A \mid E_i] \cdot Pr[E_i] = \sum_{i=1}^{n} Pr[A \cap E_i]$ (by def)

$$= \sum_{i=1}^{n} \sum_{w \in A \cap E_i} Pr[w]$$

Because E_i are disjoint, each w appears at most once

$$= \sum_{w \in A \cap (E_1 \cup E_2 \cup \ldots \cup E_n)} Pr[w]$$

$$= Pr[A \cap (E_1 \cup \ldots \cup E_n)] \qquad \square$$

Ex. Roll two die, what's the probability that the sum is 4? $Pr[A]$?

E_i = probability 1st die is i $\quad Pr[E_i] = \frac{1}{6}$ for all i

A = probability sum is 4 $\quad Pr[A \mid E_i] = \frac{1}{6}$ if $i \le 3$, $\frac{1}{6}$ if $i \ge 4$

$$Pr[A] = Pr[A \mid E_1 \cup E_2 \cup \ldots \cup E_6]$$

$$= \sum_{i=1}^{6} Pr[A \mid E_i] Pr[E_i] = \frac{1}{6} \cdot \frac{1}{6} + \frac{1}{6} \cdot \frac{1}{6} + \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{12}$$

Bayes' Rule

$$Pr[B \mid A] = \frac{Pr[A \mid B] \cdot Pr[B]}{Pr[A]}$$

also

$$Pr[B \mid A] Pr[A] = Pr[A \mid B] Pr[B] = Pr[A \cap B]$$

Independence - A and B are independent if

$$Pr[A \cap B] = Pr[A] \cdot Pr[B]$$

If $B \neq 0$, $Pr[A \mid B] = Pr[A]$

$E_1, E_2, \ldots, E_n$ are mutually independent if for every subset S of $[n]$,

$$Pr\left[\bigcap_{j \in S} E_j\right] = \prod_{j \in S} Pr[E_j]$$

Ex. $S' = \{\text{Heads}, \text{Tails}\}$, $S = \underbrace{S' \times S' \times \ldots \times S'}_{100 \text{ times}}$

E_i = event that i^{th} coin is heads

Uniform distribution,

$$Pr\left[\bigcap_{j \in T} E_j\right] = \frac{\left|\bigcap_{j \in T} E_j\right|}{|S|} = \frac{\#\text{ of ways to flip coins so that } j^{th} \text{ coin is heads for } j \in T}{|S|}$$

$$= \frac{2^{100-|S|}}{2^{100}} \text{ (product, generalized product rule)}$$

$$= \frac{1}{2^{|S|}} = \prod_{j \in T} \frac{1}{2} = \prod_{j \in T} Pr[E_j]$$

independent

Sticky coins

all heads w/ Pr $\frac{1}{2}$

all tails w/ Pr $\frac{1}{2}$

$Pr\left[\bigcap_{j \in T} E_j\right] = \frac{1}{2}$ (all heads)

$Pr[E_j] = \frac{1}{2}$ for all $j \in T$

$\prod_{j \in T} Pr[E_j] = \frac{1}{2^{|T|}}$, not independent