February 3

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Comp Sci 212

1. Introduction to Probability
2. Probability Rules
3. Conditional Probability

Announcements

- Midterm February 10, LR2
- No homework this week - practice midterm

Probability

Def. Sample space - non-empty set S

$w \in S$ outcome

subset of S - event

Ex. $S = \{1, 2, 3, 4, 5, 6\}$ $S \times S \times S \times \cdots \times S$

$S = \{\text{Heads}, \text{Tails}\}$

Def. Probability function - total function

$Pr : S \to \mathbb{R}$

- $Pr[w] \geq 0$ for all $w \in S$
- $\sum_{w \in S} Pr[w] = 1$

sum over all outcomes in S

Ex.

w	$Pr[w]$
1	$\frac{1}{6}$
2	$\frac{1}{6}$
3	$\frac{1}{6}$
4	$\frac{1}{6}$
5	$\frac{1}{6}$
6	$\frac{1}{6}$

fair die

w	$Pr[w]$
1	$\frac{1}{2}$
2	$\frac{1}{10}$
3	$\frac{1}{10}$
4	$\frac{1}{10}$
5	$\frac{1}{10}$
6	$\frac{1}{10}$

loaded die

If E is an event, $E \subseteq S$

$Pr[E] = \sum_{w \in E} Pr[w]$

Ex. $E = \{x \mid x \in [3]\}$

$Pr[E] = Pr[1] + Pr[2] + Pr[3]$

$Pr[x \leq 3]$

Ex. Uniform Distribution - $Pr[w] = \frac{1}{|S|}$ for all w

$Pr[E] = \frac{|E|}{|S|}$

Ex. Let $S' = \{\text{Heads}, \text{Tails}\}$

$S = \underbrace{S' \times S' \times S' \times \cdots \times S'}_{100 \text{ times}}$

flipping a coin 100 times

1. Uniform Distribution

$Pr[w] = \frac{1}{2^{100}}$ for all $w \in S$

$Pr[\text{At least 75 heads}] = \frac{|E|}{2^{100}}$

$E = \{$ways to flip a coin 100 times, get at least 75 heads$\}$

2. Sticky coin - not uniform

$Pr[\text{All heads}] = \frac{1}{2}$

$Pr[\text{All tails}] = \frac{1}{2}$

$Pr[\text{everything else}] = 0$

$Pr[\text{At least 75 heads}] = \frac{1}{2}$

Probability rules

- Sum rule if $E_1, \ldots, E_n$ are disjoint

$Pr[E_1 \cup E_2 \cup \cdots \cup E_n] = Pr[E_1] + Pr[E_2] + \cdots Pr[E_n]$

- Complement rule

$Pr[\underset{\text{sample space}}{S} \setminus E] = 1 - Pr[E]$

- Difference rule

$Pr[A \setminus E] = Pr[A] - Pr[A \cap E]$

- Inclusion-Exclusion

$Pr[A \cup B] = Pr[A] + Pr[B] - Pr[A \cap B]$

- Union Bound

$Pr[A \cup B] \leq Pr[A] + Pr[B]$

$Pr[A_1 \cup A_2 \cup \cdots \cup A_n] \leq Pr[A_1] + Pr[A_2] + \cdots + Pr[A_n]$

Monotonicity rule

If $A \subseteq B$, $Pr[A] \leq Pr[B]$

Union bound - 100-sided die, $S = [100]$

uniform distribution

Throw it 25 times $\underbrace{S \times S \times S \times \cdots \times S}_{25 \text{ times}}$ uniform distribution

$Pr[\text{At least 1 die roll is 1}]$

A_i the event that the i^{th} roll is 1

$= \{x \mid x \in S \times \cdots \times S, x_i = 1\}$

$Pr[A_1 \cup A_2 \cup \cdots \cup A_{25}]$

$\leq Pr[A_1] + Pr[A_2] + \cdots + Pr[A_{25}]$ (Union bound)

$= \underbrace{\frac{1}{100} + \cdots + \frac{1}{100}}_{25 \text{ times}} = \frac{1}{4}$

Conditional Probability

Roll 2 6-sided die, uniform distribution

$S = \{1, 2, 3, 4, 5, 6\}$

$S \times S$, uniform distribution

Given that the sum is 4, what is the probability that the 1st die is 1? (*)

B = set of outcomes whose sum is 4

$= \{(3,1), (2,2), (1,3)\}$

A = set of outcomes whose 1st roll is 1

$= \{(1,6), (1,5), (1,4), (1,3), (1,2), (1,1)\}$

$Pr[A \mid B] = \frac{Pr[A \cap B]}{Pr[B]} = \frac{\frac{1}{36}}{\frac{3}{36}} = \frac{1}{3}$

$A \cap B = \{(1,3)\}$

Def. Probability of A given B (Conditional Probability)

$Pr[A \mid B] = \frac{Pr[A \cap B]}{Pr[B]}$

Ex. Given that a person has two kids, one is a boy, what is the probability that both are boys?

$S = \{\text{boy}, \text{girl}\}$ $S \times S$, uniform distribution

$\{(\text{boy}, \text{boy}), (\text{girl}, \text{boy}), (\text{boy}, \text{girl}), (\text{girl}, \text{girl})\}$

$A = \{(\text{boy}, \text{boy})\}$

$B = \{(\text{boy}, \text{girl}), (\text{girl}, \text{boy}), (\text{boy}, \text{boy})\}$

$Pr[A \mid B] = \frac{Pr[A \cap B]}{Pr[B]} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$

Complement rule

$Pr[S \setminus E] = 1 - Pr[E]$

Proof. $Pr[S \setminus E] = \sum_{w \in S \setminus E} Pr[w]$

$= \underbrace{\sum_{w \in S} Pr[w]} - \sum_{w \in E} Pr[w]$

$= 1 - Pr[E]$

$\square$

$A \cap B = \{(1,3)\}$

$$\frac{\quad}{Pr[B]} = \frac{\quad}{\frac{3}{4}} = \frac{\quad}{3}$$