Comp Sci 212
Northwestern
Winter 2020 | **Lecture 11** | **January 31, 2020** | **Instructor: Shravas Rao**

In this lecture, we saw the proof of the following.

Theorem 1. *Let n be a positive even integer, and let k be any non-negative integer. Then,*

$$\binom{n}{k} = \sum_{i=0}^{k} \binom{n/2}{i}\binom{n/2}{k-i},$$

or in other words

$$\binom{n}{k} = \binom{n/2}{0}\binom{n/2}{k} + \binom{n/2}{1}\binom{n/2}{k-1} + \binom{n/2}{2}\binom{n/2}{k-2} + \cdots + \binom{n/2}{k}\binom{n/2}{0}.$$

Intuition One way to interpret this is to remember that the expression $\binom{n}{k}$ counts the number of subsets of size k of $[n] = \{1, 2, 3, \ldots, n\}$. Then the equality to expression $\sum_{i=0}^{k} \binom{n/2}{i}\binom{n/2}{k-i}$ can be thought of saying the number of subsets of size k of $[n]$ is equal to the sum over all i of the number of subsets of size i of $[n/2] = \{1, 2, \ldots, n/2\}$ times the number of subsets of size $k-i$ of $[n]\backslash[n/2] = \{n/2+1, n/2+2, \ldots, n\}$.

This makes sense, because for every subset of size k of $[n]$ we can split it into two subsets, one that's a subset of $[n/2]$ and one that's a subset of $[n]\backslash[n/2]$. The first subset can be of any size less than or equal to k, but the second must have a size so that the two add up to k.

Proof Now we'll prove Theorem 1

Proof of Theorem 1. Let $A = \{x \mid x \subseteq [n], |x| = k\}$, and $B_i = \{x \mid x \subseteq [n/2], |x| = i\}$ and $C_i = \{x \mid x \subseteq [n]\backslash[n/2], |x| = k-i\}$. We have by a theorem proved in class, $|A| = \binom{n}{k}$, $|B_i| = \binom{n/2}{i}$, and $|C_i| = \binom{n/2}{k-i}$. By the product rule, $|B_i \times C_i| = \binom{n/2}{i}\binom{n/2}{k-i}$. Because all B_i are disjoint, all $B_i \times C_i$ are disjoint, and therefore by the sum rule

$$\left|\bigcup_{i=0}^{k} B_i \times C_i\right| = \sum_{i=0}^{k} \binom{n/2}{i}\binom{n/2}{k-i}.$$

Thus, it is enough to give a bijection from A to $D = \bigcup_{i=0}^{k} B_i \times C_i$, as this implies $|A| = |D|$ by the bijection rule.

Let $f : A \to D$ be defined by $f(a) = ((a \cap [n/2], a \cap ([n]\backslash[n/2]))$. Then $f(a)$ is a pair of subsets, one of $[n/2]$ and one of $[n]\backslash[n/2]$ whose sizes add up to k, and thus $f(a)$ is in fact a member of D. To prove that f is a bijection, we need to show that $|f^{-1}(d)| = 1$ for all d in D.

Let $d = (b, c) \in D$, that is, we have a pair of subsets b and c. We must have $|b| + |c| = k$ by the definition of D. Then if $a \in f^{-1}(d)$, it must be the case that $|a| = k$. Additionally by the definition of f, $b \subseteq a$ and $c \subseteq a$, and thus $b \cup c \subseteq a$. Because b and c are disjoint, $|b \cup c| = |b| + |c| = k$, and together this implies that $b \cup c = a$, and that $f^{-1}(d) = \{a\}$. In particular, $f^{-1}(d)$ has exactly one element, the set $b \cup a$, and thus f is a bijection. $\square$

Comments The general strategy in these proofs is to come up with two sets, one whose size is the left-hand side of the equation, and one whose size is the right-hand side of the equation. Then we come up with a bijection between the two sets, which by the bijection rule proves that both sides of the equation are equal.

To come up with the two sets, you can usually replace any binomial coefficients (the things that look like $\binom{n}{k}$) with the size of a set of subsets of size k, of a set with size n. You can replace products by the size of the cartesian product by the product rule, and sums by the size of the union by the sum rule. When you use the sum rule, you should be careful to point out that your sets are disjoint.

Usually what's tricky is actually coming up with the bijection. What we noticed above is that if a is a subset of $[n]$, we can break up a into two subsets, one that's contained in $[n/2]$ and one that's contained in $[n]\backslash[n/2]$. This is what the function f does in the proof of Theorem 1.

To prove that f is in fact a bijection, we need to show that for any $d = (b, c) \in D$, $f^{-1}((b, c))$ has size exactly one. It's important to keep in mind that $f^{-1}((b, c))$ will be a set of sets, which means that $f^{-1}((b, c))$ should be a set containing just one set, but that set can have any size. To reason about $f^{-1}((b, c))$, think about how f works, and how to get to a specific pair of sets from a set of size k.