

Comp Sci 212

1. Permutations / Combinations
2. Pascal's Triangle
3. Binomial Theorem

Generalized Product - If S is a set of length k sequences $(s_1, s_2, \dots, s_k) \in S$

- n_1 = # of possibilities for s_1
 - n_2 = # of possibilities for s_2
 - n_k = # of possibilities for s_k
- $|S| = n_1 \cdot n_2 \cdot \dots \cdot n_k$

Def. Permutation a length k sequence of elements from a set of size k , no two the same.

$S = \{a, b, c, d\}$ permutations - (a, b, c, d)
 (a, b, d, c)

Thm. If S is of size k , there are $k!$ permutations. $1 \cdot 2 \cdot 3 \cdot \dots \cdot k$

Proof. Generalized Product Rule
 $n_1 = k, n_2 = k-1, \dots, n_k = 1 \rightarrow$
 # of permutations is $n_1 \cdot n_2 \cdot \dots \cdot n_k = k!$

Sorting - given a list, produce a sorted list for a sorted list of size k , - $k!$ possible original lists

Sorting algorithm must do something diff for each original list.
 permutation $\rightarrow \{0, 1\}^k$ subset of \rightarrow # of steps $\rightarrow k! \leq 2^k$
 $\Rightarrow \log k! = \Theta(k \log k)$

Division rule

Def. $f: A \rightarrow B$ is k to 1 if $|f^{-1}(b)| = k$ for every $b \in B$.

Division rule - If $f: A \rightarrow B$ is k to 1, $|A| = k \cdot |B|$.

Theorem. # of length l sequences of elements for a set S of size k , no two are the same, is $\frac{k!}{(k-l)!}$.

Proof. A = the set of permutations (length- l) of S .
 B = set of length- l sequences
 $f: A \rightarrow B$ $f(a) =$ first l elements of a .
 $f^{-1}(b) = \{a \mid b \text{ is a permutation of } b \text{ as a subsequence}\}$
 $|f^{-1}(b)| = |S \setminus b|! = (k-l)!$
 By division rule $|A| = \frac{k!}{(k-l)!}$

Theorem. # of subsets of size l of S , $|S|=k$ is $\frac{k!}{(k-l)! \cdot l!} = \binom{k}{l}$

Proof. A = set of length- l sequences from S
 B = subsets of S of size l
 $f: A \rightarrow B$, $f(a) = a$, a is now viewed as a set.

$f^{-1}(b)$ = set of permutations of b
 $|f^{-1}(b)| = l!$, by division rule $|B| = \frac{k!}{l! \cdot (k-l)!}$

of ways to buy B donuts, 3 types =
 # of $\{0, 1\}$ -strings, of length 10, 2 1's & 8 0's =
 # of subsets of size 2 of a set of size 10 = $\binom{10}{2}$

- # of ways to buy n donuts of k types =
- # of strings of length $n+k-1$, $k-1$ 1's, rest 0 =
- # of subsets of size $k-1$ of a set of size $n+k-1$ = $\binom{n+k-1}{k-1}$

of solutions to the equation $x_1 + x_2 + \dots + x_k = n$
 x_i is a non-negative integer.
 \hookrightarrow number of donuts of type i

of solutions to the equation $x_1 + \dots + x_k = nk$,
 x_i a positive integer

of solutions to $x_1 + \dots + x_{k-1} \leq n$
 x_i is a non-negative integer

Theorem $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$

Proof. Let $S = \{n+1\} = \{1, 2, 3, \dots, n+1\}$

Let A = set of subsets of S w/ k elements

A_1 = set of subsets of $S \setminus \{1\}$ with k elements

A_2 = set of subsets of $S \setminus \{1\}$ with $k-1$ elements

Enough to show that $|A| = |A_1| + |A_2|$
 $= |A_1 \cup A_2|$ - Sum rule

$f: A \rightarrow A_1 \cup A_2$

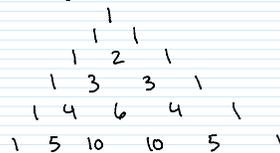
$f(x) = \begin{cases} x & \text{if } x \text{ doesn't contain } 1 \\ x \setminus \{1\} & \text{if } x \text{ does contain } 1 \end{cases}$

$f^{-1}(x) = \begin{cases} x & \text{if } |x| = k \\ x \cup \{1\} & \text{if } |x| = k-1 \end{cases}$

Therefore f is a bijection, $|A| = |A_1 \cup A_2|$
 \hookrightarrow bijection rule \square

Def. $\binom{n}{k} = 0$ if $k < 0$ or $k > n$

Pascal's Triangle



n th row, k th column is $\binom{n}{k}$