January 27

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Comp Sci 212

1. Permutations / Combinations
2. Pascal's Triangle
3. Binomial Theorem

Generalized Product - If S is a set of length-k sequences $(s_1, s_2, \dots s_k) \in S$

- n_1 = # of possibilities for s_1
- n_2 = # of possibilities for s_2

⋮

- n_k = # of possibilities for s_k

$|S| = n_1 \cdot n_2 \cdots n_k$

Def. Permutation a length k sequence of elements from a set of size k, no two the same.

$S = \{a, b, c, d\}$ permutations - (c, b, a, d)
- (a, b, c, d)

Thm. If S is of size k, there are $k!$ permutations. $1 \cdot 2 \cdot 3 \cdot 4 \cdots k$

Proof. Generalized Product Rule

$n_1 = k,\ n_2 = k-1,\ \dots\ n_k = 1 \rightarrow$

# of permutations is $n_1 \cdot n_2 \cdots n_k = k!$

Sorting - given a list, produce a sorted list.

For a sorted list of size k, - $k!$ possible original lists

Sorting algorithm must do something diff for each original list.

permutation $\longrightarrow \{0,1\}^{\ell}$ → # of steps
subset of injection

$\rightarrow k! \le 2^{\ell}$

$\ell \ge \log k! = \Theta(k \log k)$

Division rule

Def. $f: A \to B$ is k to 1 if $|f^{-1}(b)| = k$ for every $b \in B$. $\{a \mid a \in A, f(a) = b\}$

Division rule - If $f: A \to B$ is k to 1,
$|A| = k \cdot |B|$.

Theorem. # of length ℓ sequences of elements for a set S of size k, no two are the same, is $\frac{k!}{(k-\ell)!}$

Proof. A = the set of permutations (length-k) of S.
B = set of length-ℓ sequences

$f: A \to B$, $f(a)$ = first ℓ elements of a.

$f^{-1}(b) = \{ b \circ c \mid c$ is a permutation of $S \setminus b \}$ (concatenation; $S \setminus b$ → view this as a set)

$|f^{-1}(b)| = |S \setminus b|! = (k-\ell)!$

By division rule $|B| = \frac{k!}{(k-\ell)!}$ □

Theorem. # of subsets of size ℓ of S, $|S| = k$ is $\frac{k!}{(k-\ell)!\,\ell!} = \binom{k}{\ell}$

Proof. A = set of length-ℓ sequences from S
B = subsets of S of size ℓ

$f: A \to B$, $f(a) = a$, a is now viewed as a set.

$f^{-1}(b)$ = set of permutations of b

$|f^{-1}(b)| = \ell!$, by division rule $|B| = \frac{k!}{\ell!\,(k-\ell)!}$ □

# of ways to buy 8 donuts, 3 types =
of $\{0,1\}$-strings, of length 10, 2 1's 8 0's =
of subsets of size 2 of a set of size 10 = $\binom{10}{2}$

# of ways to buy n donuts of k types =
# of strings of length $n+k-1$, $k-1$ 1's, rest 0 =
# of subsets of size $k-1$ of a set of size $n+k-1 = \binom{n+k-1}{k-1}$

# of solutions to the equation =
$x_1 + x_2 + \cdots x_k = n$
x_i is a non-negative integer.
↳ number of donuts of type i

# of solutions to the equation =
$x_1 \cdots x_k = n+k$,
x_i a positive integer

# of solutions to =
$x_1 + \cdots x_{k-1} \le n$
x_i is a non-negative integer

Theorem $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$

Proof. Let $S = [n+1] = \{1, 2, 3, \dots, n+1\}$
Let A = set of subsets of S w/ k elements
A_1 = set of subsets of $S \setminus \{1\}$ with k elements
A_2 = set of subsets of $S \setminus \{1\}$ with $k-1$ elements.

Enough to show that $|A| = |A_1| + |A_2|$
$= |A_1 \cup A_2|$ - Sum rule

$f: A \to A_1 \cup A_2$

$f(x) = \begin{cases} x & \text{if } x \text{ doesn't contain } 1 \\ x \setminus \{1\} & \text{if } x \text{ does contain } 1 \end{cases}$

$f^{-1}(x) = \begin{cases} x & \text{if } |x| = k \\ x \cup \{1\} & \text{if } |x| = k-1 \end{cases}$

Therefore f is a bijection, $|A| = |A_1 \cup A_2|$
↳ bijection rule □

Def. $\binom{n}{k} = 0$ if $k < 0$ or $k > n$

Pascal's Triangle

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

n^{th} row, k^{th} column is $\binom{n}{k}$