January 22

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Comp Sci 212

1. Relations / Functions
2. Asymptotics

Relation - subset of $A \times B$ from A to B

Function - relation for every $a \in A$, appears at most <u>once</u> in a pair.

$\{(a, b) \mid a \in [n], b \in [2n], b = 2a\} \subseteq [n] \times [2n]$

Also a function, $f: [n] \to [2n]$, $f(a) = 2a$

$G = \{(a, b) \mid a \in [n], b \in \{\text{even}, \text{odd}\}, a \text{ is "}b\text{"}\}$

$g: [n] \to \{\text{even}, \text{odd}\}$, $g(a) = \begin{cases} \text{odd} & \text{if } a \text{ is odd} \\ \text{even} & \text{if } a \text{ is even} \end{cases}$

$(1, \text{odd}) \in G$ $(1, \text{even}) \notin G$

$[n] = \{1, 2, 3, \ldots, n\}$

Injective if each $b \in B$ appears at most once ($|f^{-1}(b)| \le 1$ for all $b \in B$) ($f(a) = f(b) \to a = b$)

Surjective if each $b \in B$ appears at least once ($|f^{-1}(b)| \ge 1$ for all $b \in B$)

f is injective, $f(a) = f(b) \to 2a = 2b \to a = b$

not surjective, $f^{-1}(1) = \{a \mid f(a) = 1\} = \phi$

g is surjective if $n \ge 2$

$g^{-1}(\text{even}) = \{a \mid a \in [n] \text{ and } a \text{ is even}\}$

$g^{-1}(\text{odd}) = \{a \mid a \in [n] \text{ and } a \text{ is odd}\}$

not injective if $n \ge 3$

If $f: A \to B$ is an injective function, $|B| \ge |A|$

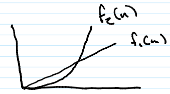
total, (every a appears at least once)	surjective	$\lvert A\rvert \ge \lvert B\rvert$
	bijective	$\lvert A\rvert = \lvert B\rvert$

Asymptotics

Question - what is the behavior of a function $f(n): \mathbb{R} \to \mathbb{R}$ as n becomes large?

$f_1(n) = 100 n \log n$ $f_2(n) = 2n^2 - n$

How do we compare two functions?



Def. $f(n) = O(g(n))$ if $\limsup_{n\to\infty} \left|\frac{f(x)}{g(x)}\right| < \infty$

$f(n) = o(g(n))$ if $\limsup_{n\to\infty} \left|\frac{f(x)}{g(x)}\right| = 0$

$f(n) = \Omega(g(n))$ if $g(n) = O(f(n))$

$f(n) = \omega(g(n))$ if $g(n) = o(f(n))$

$f(n) = \Theta(g(n))$ if $f = O(g(n)), \Omega(g(n))$

Ex. $2n^2 + n = \Theta(n^2)$ $\lim_{n\to\infty} \frac{2n^2+n}{n^2} = 2$

$n = o(n^2)$, $\lim_{n\to\infty} \frac{n}{n^2} = 0$

$1 = \omega\left(\frac{1}{\log n}\right)$, $\frac{1}{\log(n)} = o(1)$, $\lim_{n\to\infty} \frac{1}{\log n} = 0$

- Ignore constant factors
- Ignore lower order terms

Ex. $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 = \Theta(g(x))$, $g(x) = x^n$

<u>Fact</u> If $a < b$, then $\lim_{n\to\infty} \frac{n^a}{n^b} = 0$, so $n^a = O(n^b)$, $o(n^b)$

Theorem. For $a, b > 0$, $\lim_{n\to\infty} \frac{\log^a(n)}{n^b} = 0$

$(\log(n))^{10000000} = o(n^{0.000001})$

Proof. Assume $a = 1$

By L'Hôpital's rule,

$\lim_{n\to\infty} \frac{\log(n)}{n^b} = \lim_{n\to\infty} \frac{\frac{1}{n}}{b \cdot n^{b-1}} = \lim_{n\to\infty} \frac{1}{b \cdot n^b} = 0.$

If $a \ne 1$, $\left(\lim_{n\to\infty} \frac{\log(n)}{n^{\frac{b}{a}}}\right) = 0$, take both sides to the a^{th} power $\square$

$2^n = o(4^n)$, $\lim_{n\to\infty} \frac{2^n}{4^n} = \lim_{n\to\infty} \frac{1}{2^n} = 0$

$1 = \Theta(10000000)$

Theorem. If $\log(f(n)) = o(\log(g(n)))$, ← only for little o $f(n), g(n) > 0$, then $f(n) = o(g(n))$

Proof. If $\log(f(n)) = o(\log(g(n)))$ then

$\lim_{n\to\infty} \frac{\log(f(n))}{\log(g(n))} = 0$,

$\lim_{n\to\infty} \log(f(n)) - \log(g(n)) = -\infty$

$0 = \lim_{n\to\infty} e^{\log(f(n)) - \log(g(n))} = \lim_{n\to\infty} \frac{f(n)}{g(n)}$ $\square$

Ex. $f(n) = (\log(\log(n)))^{\log(n)}$ $g(n) = n^2$

$\log f(n) = \log(n) \cdot \log(\log(\log(n)))$ → $\log(g(n)) = o(\log(f(n)))$ ↓ $g(n) = o(f(n))$

$\log(g(n)) = 2 \cdot \log(n)$

$\lim_{n\to\infty} \frac{\log(g(n))}{\log(f(n))} = \lim_{n\to\infty} = \frac{2}{\log(\log(\log(n)))}$

$= 0$