January 13

Monday, January 13, 2020 3:18 PM

Comp Sci 212

1. More Induction
2. Faulty Induction
3. Strong Induction

Induction

- Base Case $P(0)$
- Inductive step $P(n) \to P(n+1)$
- $P(n)$ is true for all n

Fibonacci Numbers

$F_0 = 0$ $F_1 = 1$ $F_n = F_{n-1} + F_{n-2}$ for all non-negative integers $n \geq 2$

$F_2 = F_1 + F_0 = 1$ $F_5 = 5$

$F_3 = 2$ $F_6 = 8$

$F_4 = 3$ $F_7 = 13$

Thm. For all non-negative integers n

$$F_0 + F_1 + F_2 + \cdots + F_n = F_{n+2} - 1$$

Ex. $F_0 = 0$ $F_2 - 1 = 1 - 1 = 0$

$F_0 + F_1 = 1$ $F_3 - 1 = 2 - 1 = 1$

$F_0 + F_1 + F_2 = 2$ $F_4 - 1 = 3 - 1 = 2$

$F_0 + F_1 + F_2 + F_3 = 4$ $F_5 - 1 = 5 - 1 = 4$

Proof. Use Induction

Base Case $F_0 = 0$, $F_2 - 1 = 1 - 1 = 0$ ✓

Inductive step - Assume $F_0 + F_1 + \cdots + F_n = F_{n+2} - 1$

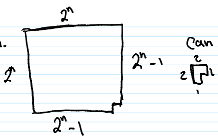
$$\underbrace{F_0 + F_1 + F_2 + \cdots + F_n}_{F_{n+2} - 1} + F_{n+1} = F_{n+2} + F_{n+1} - 1$$

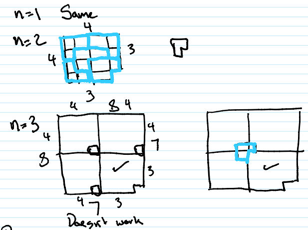
$$= F_{n+3} - 1$$

□

Friday - $1 + 2 + \cdots + n$

$S_n = 1 + \cdots + n$ $S_{n+1} = S_n + (n+1)$

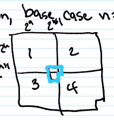
Thm.  can be tiled by



Proof. Use induction, base case $n=1$, shapes are the same.

Inductive step

Assume true for n



Divide regions as in figure. Tile 4, using solution for n.

Tile 1 using assumption, leave out bottom-right,

2 → bottom left 3 → top right, place in remaining portion

Faulty Induction

Thm. In any group of n people, everyone has the same name.

Proof. Use Induction, base case $n=1$ obvious.

Inductive step - Assume true for n

Number everyone from 1 to $n+1$.

$\underbrace{1 \quad 2 \quad 3 \quad 4 \quad \cdots \quad n}_{\text{Group 1}}$ $n+1$

$1 \quad \underbrace{2 \quad 3 \quad 4 \quad \cdots \quad n \quad n+1}_{\text{Group 2}}$

Everyone in group 1 has the same name, so does everyone in group 2. Therefore, everyone has the same name. □

> If $n+1 = 2$

$\underbrace{1}_{\text{Group 1}} \quad \underbrace{2}_{\text{Group 2}}$ No overlap

$P(1)$ does not imply $P(2)$

$P(n)$ implies $P(n+1)$ if $n \geq 2$.

Strong Induction

- Prove $P(0)$ - Base case
- If $P(0), P(1), \ldots, P(n) \to P(n+1)$ - Inductive step.
- Then $P(n)$ is true for all n.
- multiple base cases, base case not $n=0$

Theorem - Every non-negative integer $n \geq 2$ $(n \in \mathbb{N})$ can be written as a product of prime numbers.

↳ if it's only divisible by 1 & itself.

Ex. $n = 5$ $n = 15 = 5 \cdot 3$

$n = 7$ $n = 24 = 3 \cdot 2 \cdot 2 \cdot 2$

Proof. Use strong induction. Base case $n = 2$, is prime.

Inductive step - assume true for all numbers up to n.

Case 1: $n+1$ is prime

Case 2: $n+1$ is not prime. Then it's divisible by a, $2 \leq a < n+1$.

$n+1 = a\left(\frac{n+1}{a}\right)$, and a, $\frac{n+1}{a}$ are integers at least 2, and can be written as a product of primes.

Therefore $n+1$ can be written as a product of primes. □

Theorem. If n is an integer and $n \geq 4$, then $2^n \leq 1 \cdot 2 \cdot 3 \cdots n$ $(n!)$.

n	2^n	$n!$
1	2	1
2	4	2
3	8	6
4	16	24
5	32	120
6	64	720

Proof. Use Induction.

Base Case $n = 4$, $2^n = 16 \leq n! = 24$

Inductive step assume true for n, $2^n \leq n!$.

$2 \leq (n+1)$, therefore $2^n \cdot 2 \leq n! \cdot (n+1)$,

that is $2^{n+1} \leq (n+1)!$

□